Solutions on Class Exercises

1. Show that $n! < n^n$ whenever $n$ is a positive integer greater than 1.
   
   Proof: Base case. When $n = 2$, $n! = 2 \times 1 = 2$ and $n^n = 2^2 = 4$. Clearly, $2 < 4$.
   
   Inductive step. Assume that $k! < k^k$. Let us now show that $(k + 1)! < (k + 1)^{(k+1)}$.
   
   
   $$(k + 1)! = k!(k + 1) < k^k(k + 1) \text{ by our assumption}$$
   
   $$< (k + 1)^k(k + 1)$$
   
   $$= (k + 1)^{(k+1)}.$$
   
   Therefore, by induction, $n! < n^n$ whenever $n$ is a positive integer greater than 1.

2. Show that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   
   Proof: Base case. When $n = 1$, $1^2 + 2^2 + \ldots + n^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$.
   
   Hence, the statement is true when $n = 1$.
   
   Inductive step. Assume that $1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$. Let us show that $1^2 + 2^2 + \ldots + (k + 1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.
   
   $$1^2 + 2^2 + \ldots + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \text{ by our assumption}$$
   
   $$= (k + 1)\frac{k(2k + 1) + 6(k + 1)}{6}$$
   
   $$= (k + 1)\frac{2k^2 + 7k + 6}{6}$$
   
   $$= (k + 1)\frac{(2k + 3)(k + 2)}{6}$$
   
   $$= \frac{(k + 1)(k + 2)(2(k + 1)+1)}{6}.$$
   
   Therefore, by induction, $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$. 