The Minimum Guarding Tree Problem

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September 3, 2013

Abstract

Given a set $\mathcal{L}$ of non-parallel lines in the plane and a nonempty subset $\mathcal{L}' \subseteq \mathcal{L}$, a guarding tree for $\mathcal{L}'$ is a tree contained in the union of the lines in $\mathcal{L}$ such that if a mobile guard (agent) runs on the edges of the tree, all lines in $\mathcal{L}'$ are visited by the guard. Similarly, given a connected arrangement $\mathcal{S}$ of line segments in the plane and a nonempty subset $\mathcal{S}' \subseteq \mathcal{S}$, we define a guarding tree for $\mathcal{S}'$. The minimum guarding tree problem for a given set of lines or line segments is to find a minimum-length guarding tree for the input set.

We provide a simple alternative (to [29]) proof of NP-hardness of the problem of finding a guarding tree of minimum length for a set of orthogonal (axis-parallel) line segments in the plane. Then, we present two approximation algorithms with factors 2 and 3.98, respectively, for computing a minimum guarding tree for a subset of a set of n arbitrary non-parallel lines in the plane; their running times are $O(n^8)$ and $O(n^6 \log n)$, respectively. Finally, we show that this problem is NP-hard for lines in 3-space.

Keywords: guarding tree, NP-hardness, approximation algorithm.

AMS subject classifications: 68W25, 68R10, 68U05.

1 Introduction

A set $\mathcal{X}$ of lines or line segments in the plane (or the underlying arrangement $\mathcal{A}(\mathcal{X})$) is said to be *connected* if there exists a path $\xi \subset \cup_{l \in \mathcal{X}} l$ from any point $p' \in l'$ to any other point $p'' \in l''$, for any $l', l'' \in \mathcal{X}$. Observe that an arrangement of lines is connected if and only if not all lines are parallel, i.e., there exist two non-parallel lines in $\mathcal{X}$.

A connected set of lines or line segments can model the corridors in a building. Consider a mobile guard (agent) that patrols the corridors; we assume that the guard has unlimited visibility, so that all points of a line/segment can be "seen" in both directions from any point on it. The problem is to find a shortest "guarding network" for the agent, e.g., a path, a tree, or a closed route, such that when traversing all the edges of the network (moving only within the network), each line/segment is visited at least once, thereby allowing the mobile guard to see all points of all lines/segments of the input.

In this paper we study the variant of the problem in which the guarding network is restricted to form a tree; obviously, a minimum-length (connected) guarding network is always a tree. Let $\mathcal{X}$ be a connected set of lines or line segments in the plane, and let $\mathcal{X}' \subseteq \mathcal{X}$, $\mathcal{X}' \neq \emptyset$. A *guarding*

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tree $T = (V(T), E(T))$ for $\mathcal{X}'$ is a tree with vertices (points) and edges (segments) contained in the union of the elements in $\mathcal{X}$, such that if a mobile agent traverses the edges of T, all elements in $\mathcal{X}'$ are visited by the agent[1]. The *length* $|T|$ of a guarding tree T is the sum of the weights of its edges, where the weight of an edge is its Euclidean length. Two natural optimization problems are:

The minimum guarding tree problem for lines (MGTL)

> Given a connected set $\mathcal{L}$ of lines in the plane and a nonempty subset $\mathcal{L}' \subseteq \mathcal{L}$, find a minimum-length guarding tree for $\mathcal{L}'$.

The minimum guarding tree problem for segments (MGTS)

> Given a connected set $\mathcal{S}$ of line segments in the plane and a subset $\mathcal{S}' \subseteq \mathcal{S}$, find a minimum-length guarding tree for $\mathcal{S}'$.

Previously, MGTS appears to have been only considered for arrangements of axis-parallel segments, so-called *grids*: Xu and Brass [29, 30] proved the NP-hardness of MGTS using a reduction from the connected vertex cover problem in planar graphs with maximum degree four.

Our results. In Section 2 we show that MGTS is NP-hard, and in view of [29, 30], we reprove this result in a simpler way. In Sections 3 and 4 we present two approximation algorithms for MGTL. The first algorithm achieves ratio 2 and runs in $O(n^8)$ time, while the second algorithm achieves ratio 3.98 and runs in $O(n^6 \log n)$ time. In Section 5 we show that the problem of computing a minimum-length guarding tree for a connected arrangement of lines in 3-space is NP-hard even for orthogonal lines.

We emphasize that the planar hardness result is for line segments, and since MGTS is closely related to the group Steiner tree problem, by [10, 12] one immediately obtains an approximation randomized algorithm with ratio $O(\log^3 n)$ for MGTS [29]. On the other hand, no hardness results in known for lines in the plane, and here we obtain two approximation algorithms with constant ratio for MGTL.

Related work. In the *minimum watchman route problem*, the objective is the same as in MGTL or MGTS, but the guarding network is restricted to form a *route* (closed tour). This variant has been studied in [9, 29, 30, 20]. Xu and Brass [29, 30] proved the NP-hardness of this variant for axis-parallel line segments, which was later reproved in a simpler way by Dumitrescu et al. [9]; in the latter paper [9], the minimum watchman route problem for lines is shown to be polynomially tractable. Recently, Mitchell [20] revisited the minimum watchman route problem and gave a polynomial time algorithm for half-lines and a $O(\log^2 n)$-approximation algorithm for line segments (and more generally for the watchman route problem in polygons with holes).

The MGT problem for lines or segments in the plane is a variant of the minimum corridor connection problem (MCC) [8, 17], which itself is a variant of the face-spanning subgraph problem [21]. Given an orthogonal polygon partitioned into orthogonal subpolygons (rooms), the objective of MCC is to find a minimum-length tree lying along its boundary and/or the boundary of the rooms that includes at least one point from the polygon boundary and at least one point from the boundary of each of the rooms. The decision version of MCC as well as its variant MCC-R, where rooms are rectangles, are strongly NP-complete [2, 14]. Some approximation algorithms for MCC-R and its variant MCC_k, where rooms are rectilinear polygons with at most k sides (k is a constant), have been proposed in [15].

[1] Alternatively, we may refer to a guarding tree as a *watchman tree* and to the minimum guarding tree problem as the *minimum watchman tree problem*.

A wider perspective locates the MGT problem as a variant of the art gallery problem for segments. Related work includes [3, 4, 5, 13, 18, 22, 23, 26, 27]; see also the survey articles [19, 25, 28]. For instance, Bose et al. [3] consider the problem of guarding the cells of a line arrangement when the lines act as obstacles. Further, the MGT problem for lines/segments is closely related to the group Steiner tree problem introduced by Reich and Widmayer [24]. By applying the $O(\log^3 n)$ approximation randomized algorithm for group Steiner tree due to Garg et al. [12] as further refined by Fakcharoenphol et al. [10], we immediately obtain an approximation algorithm with the same ratio for both MGTL and MGTS [29]. Note that the above mentioned face-spanning subgraph problem [21] and its generalization [7] (in which the instance is a planar embedded edge-weighted graph G and a subset F of faces of G, and the problem is to find a minimum tree T such that the vertex set of T contains at least one vertex from the boundary of each face in F) also belong to the family of group Steiner tree problems.

Definitions and notations. Given a set of lines $\mathcal{L}$, let $V(\mathcal{A}(\mathcal{L}))$ denote the set of vertices of the corresponding arrangement $\mathcal{A}(\mathcal{L})$. Let $G(\mathcal{L})$ be the weighted planar graph with vertex set $V(\mathcal{A}(\mathcal{L}))$ whose edges connect successive vertices on the lines in $\mathcal{L}$; the weight of an edge is the Euclidean distance between the corresponding vertices along the connecting line; $G(\mathcal{L})$ has $O(n^2)$ vertices and $O(n^2)$ edges. For $s, t \in V(\mathcal{A}(\mathcal{L}))$, let $\pi(s,t)$ denote a shortest path connecting s and t in $G(\mathcal{L})$, and let $|\pi(s,t)|$ denote its length. For a subset $\mathcal{L}' \subseteq \mathcal{L}$, let $T_{\text{opt}}(\mathcal{L}')$ denote an optimal guarding tree for $\mathcal{L}'$, and for two points p and q in the plane, let $|pq|$ denote the Euclidean length of the line segment pq.

A *weakly simple polygon* is a simply connected subset of the plane whose boundary is the union of a finite number of line segments. A polygonal route (cycle) $\mathcal{R}$ is said to be *noncrossing* if $\mathcal{R}$ is the cyclical sequence of line segments in a boundary traversal of a weakly simple polygon. If π is a path connecting two vertices in $G(\mathcal{L})$, π^R denotes the same path traversed in the *reverse* (opposite) order. Analogous to guarding routes for an arrangement of lines, a *guarding route* (or just *route*) for a subset of vertices $X \subseteq V(\mathcal{A}(\mathcal{L}))$ is any closed (polygonal) curve contained in $\mathcal{A}(\mathcal{L})$ that visits all vertices in X.

2 Line segments in the plane: NP-hardness

In this section we present a simple alternative proof of NP-hardness of MGTS for axis-parallel segments in the plane. Xu and Brass proved the NP-hardness of MGTS by a reduction from the connected vertex cover problem in planar graphs with maximum degree four [29, 30]. Our simpler alternative proof is a reduction from rectilinear Steiner tree (RST). The same construction was used in the NP-hardness proof of the minimum watchman route problem for orthogonal line segments in [9].

The rectilinear Steiner tree problem [11] is known to be NP-hard; its decision version can be formulated as follows.

The rectilinear Steiner tree problem (RST)

Given a set S of n lattice points in the plane, and a positive integer m, does there exists a rectilinear Steiner tree for S of total length at most m?

Theorem 1. [29, 30] *MGTS is NP-hard even for orthogonal segments.*

Proof. Let P be a set of n lattice points, and let R be the minimal axis-parallel rectangle containing P. Consider the arrangement $H(P)$ of axis-parallel lines induced by P, the so called Hanan grid [16], and add to an initially empty set $\mathcal{S}$ the line segments that are the intersections of R with the lines of $H(P)$. Next, for each point $p \in P$, we add to $\mathcal{S}$ a short horizontal line segment $s(p)$ of length $\frac{1}{10n}$ at vertical distance $\frac{1}{10n}$ from p inside R. Refer to Fig. 1 for an illustration. Observe that $s(p)$ can be only visited from the grid segment incident to p. The reduction, and thereby the NP-hardness of MGTS, follows by the following claim.

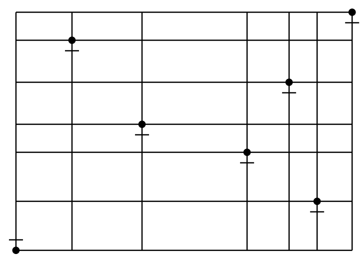

Figure 1: NP-hardness reduction [9].

Claim. *For a positive integer m, there exists a rectilinear Steiner tree for P of length at most m if and only if there exists a guarding tree for $\mathcal{S}$ of length at most $m + 0.1$.*

Proof of Claim. Let m be a postive integer and assume that there exists a rectilinear Steiner tree for P of length at most m. By a result of Hanan, $H(P)$ is known to contain a minimum (integer) length rectilinear Steiner tree T_{opt} for P [16]; thus, T_{opt} is contained in the arrangement $\mathcal{A}(\mathcal{S})$ made by the segments in $\mathcal{S}$ and its length is at most m. Convert T_{opt} into a guarding tree for $\mathcal{S}$ by augmenting it with at most n edges of length $\frac{1}{10n}$ each, as needed. The total cost of augmentation does not exceed $1/10$, and we have a guarding tree for $\mathcal{S}$ of length at most $m + 0.1$, as required.

Conversely, now assume that there exists a guarding tree for $\mathcal{S}$ of length at most $m + 0.1$, for a positive integer m. We can convert the guarding tree into a rectilinear Steiner tree for P by augmenting it with at most n edges each of length $\frac{1}{10n}$, as needed. The total cost of augmentation does not exceed $1/10$, and we have now a rectilinear Steiner tree T for P of length at most $m+0.2$. The tree T is contained in $\mathcal{A}(\mathcal{S})$, and, by shortcutting all subsegments (if any) of $s(p)$, we can assume that T is contained in $H(P)$. We thereby have a rectilinear Steiner tree T for P of length at most $m + 0.2$ contained in $H(P)$. Then, by Hanan's result, $H(P)$ contains a rectilinear Steiner tree T for P of integer length at most $m + 0.2$, and thus of length at most m, as required.

This concludes the proof of Claim and thereby the proof of Theorem 1. $\square$

3 Lines in the plane: A slower 2-approximation

While the computational complexity of MGTL remains unsettled, in this section we obtain the following approximation result.

Theorem 2. *There exists a ratio 2 approximation algorithm for MGTL, running in $O(n^8)$ time.*

Proof. Let $\mathcal{L}$ be a connected arrangement of n lines in the plane and let $\mathcal{L}' \subseteq \mathcal{L}$ be a non-empty subset of $\mathcal{L}$. In [9], the authors provide an algorithm for computing a shortest guarding route (i.e., closed tour) $\mathcal{R}_{\text{opt}}$ for $\mathcal{L}'$ in $O(n^8)$ time. Some edges in $\mathcal{R}_{\text{opt}}$ may be traversed twice; we take all edges in $\mathcal{R}_{\text{opt}}$ and remove "multiplicities", if any; the resulting graph is $\Pi \subseteq G(\mathcal{L})$. (The graph $\Pi \subseteq G(\mathcal{L})$ is the union of all the edges in $\mathcal{R}_{\text{opt}}$.) Since by the claim below $\mathcal{R}_{\text{opt}}$ consists of $O(n^2)$ edges, Π has $O(n^2)$ edges and can be computed in $O(n^2)$ time from $\mathcal{R}_{\text{opt}}$.

Claim. $\mathcal{R}_{\text{opt}}$ *consists of* $O(n^2)$ *edges.*

Let $S = \{x_1, x_2, \ldots, x_k\} \subseteq V(\mathcal{A}(\mathcal{L}))$ be a minimal subset of vertices of $\mathcal{R}_{\text{opt}}$ that covers/sees all lines; assume that when moving along $\mathcal{R}_{\text{opt}}$, the order we visit vertices of S for the first time is $x_1, x_2, \ldots, x_k$. Since each element $x \in S$ covers at least one line that is uncovered by elements in $S \setminus \{x\}$, we have $k \leq n$.

Let π_i^* be the path between x_i and x_{i+1} in $\mathcal{R}_{\text{opt}}$, for $i = 1, \ldots, k$, with $x_{k+1} = x_1$. Obviously, π_i^* is a shortest path, since otherwise, $\mathcal{R}_{\text{opt}}$ would not be optimal. Since π_i^* is a shortest path between x_i and x_{i+1}, the intersection $\pi_i^* \cap \ell$ for any line $\ell \in \mathcal{L}$ is a single line segment (otherwise, π_i^* would not be the shortest). Consequently, π_i^* consists of at most n line segments, and since $k \leq n$, $\mathcal{R}_{\text{opt}}$ has $O(n^2)$ segments, which proves the claim.

Next, let T be a minimum spanning tree of Π; T can be computed in $O(n^2 \log n)$ time [6]. Now, by construction, T is a guarding tree for $\mathcal{L}'$, and $|T| \leq |\mathcal{R}_{\text{opt}}|$. Since doubling the edges of an optimal guarding tree T_{opt} for $\mathcal{L}'$ results in a guarding route for $\mathcal{L}'$, we have $|\mathcal{R}_{\text{opt}}| \leq 2 \cdot |T_{\text{opt}}|$, and thus $|T| \leq |\mathcal{R}_{\text{opt}}| \leq 2 \cdot |T_{\text{opt}}|$, as required. □

4 Lines in the plane: A slightly faster 3.98-approximation

The running time of the algorithm from Section 3 is quite high. In this section, we present a slightly faster algorithm for MGTL, with running time $O(n^6 \log n)$ and approximation ratio 3.98.

Theorem 3. *There exists a ratio* 3.98 *approximation algorithm for MGTL, running in* $O(n^6 \log n)$ *time.*

Preprocessing step. Let $\mathcal{L}$ be a set of non-parallel lines in the plane and let $\mathcal{L}' \subseteq \mathcal{L}$ be a non-empty subset of $\mathcal{L}$. First, we handle, in $O(n)$ time, the trivial case in which all lines in $\mathcal{L}'$ are incident to one point. Assume now that not all lines in $\mathcal{L}'$ are incident to one point. We then find a suitable coordinate system such that, with respect to this system, no line in $\mathcal{L}$ and no line through two vertices in $V(\mathcal{A}(\mathcal{L}))$ is axis-parallel or makes an angle of 60 degrees with the horizontal axis. Compute all-pairs shortest-paths $\pi(s,t)$ between vertices in the graph $G(\mathcal{L})$, and record their lengths for future use. By using the Floyd-Warshall algorithm, this takes $O(|V(\mathcal{A}(\mathcal{L}))|^3) = O(n^6)$ time [6], where $n = |\mathcal{L}|$; note that $G(\mathcal{L})$ is a planar graph that can be constructed in $O(n^2)$ time [1].

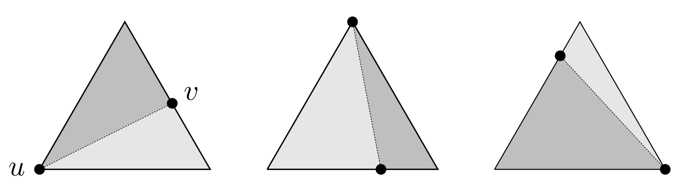

Figure 2: The equilateral triangle $\Delta(u,v)$ and its two corner triangles (lightgray and gray) defined by u and v.

For each pair of vertices in $u, v \in V(\mathcal{A}(\mathcal{L}))$ consider the smallest equilateral triangle $\Delta = \Delta(u,v)$ that includes u, v and whose lower side is horizontal; see Fig. 2. (Notice that either u or v is a vertex of Δ, and both u and v lie on the boundary of Δ.) There are $O(n^4)$ pairs of vertices and, thus, also $O(n^4)$ equilateral triangles. Each such triangle is the union of two *corner triangles* defined by u and v. For each corner triangle, we compute an "extremal line" ℓ associated with it (if it exists); "extremal lines" are defined further below (in the paragraph "Corners and extremal lines") and can be readily computed in $O(n)$ time by a simple incremental algorithm. We then determine a vertex $w \in \ell \cap \Delta$, if it exists, that minimizes $|\pi(w,u)| + |\pi(w,v)|$. Since we have recorded all-pairs shortest paths, this also takes $O(n)$ time. Hence the extremal lines and minimizing vertices on these lines in the relevant corner triangles can be determined in $O(n^5)$ time over all pairs of vertices $u, v \in V(\mathcal{A}(\mathcal{L}))$.

Besides computing shortest paths and determining extremal lines, we set up a structure that allows fast answers to the following queries: Given a query equilateral triangle E with a horizontal lower side, does it intersect each line in $\mathcal{L}'$? Let $\mathcal{L}' = \mathcal{L}^+ \cup \mathcal{L}^-$ be the partition of the lines in $\mathcal{L}'$ into lines of positive and negative slopes, respectively. Next, let $\mathcal{L}^+ = \mathcal{L}_1^+ \cup \mathcal{L}_2^+$ be the partition of the lines of positive slope into lines of slopes at most $\sqrt{3}$ and at least $\sqrt{3}$, respectively; analogously, let $\mathcal{L}^- = \mathcal{L}_3^- \cup \mathcal{L}_4^-$ be the partition of the lines of negative slope into lines of slopes at most $-\sqrt{3}$ and at least $-\sqrt{3}$, respectively. Let $E = r_1 r_2 r_3$ be labeled counterclockwise starting with its lower right corner. Observe that E misses some line in $\mathcal{L}'$ if and only if there exists a line in $\mathcal{L}^+$ below r_1, or a line in $\mathcal{L}_1^+$ above r_2, or a line in $\mathcal{L}_2^+$ above r_3, or a line in $\mathcal{L}^-$ below r_3, or a line in $\mathcal{L}_3^-$ above r_1, or a line in $\mathcal{L}_4^-$ above r_2. To answer such queries efficiently, we preprocess the lines in $\mathcal{L}'$ by using the point-line duality transform [1, Ch. 8.2].

The duality transform D maps a point $p = (a, b)$ to the non-vertical line p^* with equation $y = ax - b$. Conversely, a non-vertical line ℓ with equation $y = ax + b$ is mapped to the point $\ell^* = (a, -b)$. It is well known that D is order preserving: p lies above ℓ if and only if ℓ^* lies above p^*. Let $P^+ = \{\ell^* \mid \ell \in \mathcal{L}^+\}$. Then there exists a line in $\mathcal{L}^+$ below r_1 if and only if there exists a point in P^+ above the line r_1^*; equivalently, the line r_1^* either intersects $\mathrm{conv}(P^+)$ or lies below it. In the preprocessing step, we compute $\mathrm{conv}(P^+)$ and then use a binary search that allows answering this test in $O(\log n)$ time. The other five questions relevant in checking whether E misses some line in $\mathcal{L}'$ can be rephrased similarly, using duality; this allows answering a query intersection test in $O(\log n)$ time.

Guessing key elements of an optimal guarding tree. Consider now a minimum-length guarding tree $T_{\mathrm{opt}} = T_{\mathrm{opt}}(\mathcal{L}')$ for $\mathcal{L}'$. Let E be a minimal equilateral triangle containing T_{opt}, with a horizontal lower side. Since T_{opt} visits all lines in $\mathcal{L}'$, E intersects all lines in $\mathcal{L}'$. Let $A = \{a_1, a_2, a_3\} \subseteq V(\mathcal{A}(\mathcal{L}))$ be the set of at most three vertices of $\mathcal{A}(\mathcal{L})$ that determine E, labeled counterclockwise starting with the lower side of E; we have $2 \leq |A| \leq 3$. In particular, T_{opt} visits A. Let $E = r_1 r_2 r_3$ be labeled counterclockwise starting with its lower right corner (see Fig. 3). Suppose we *guess* A, hence E is determined, and then we *mark* all lines in $\mathcal{L}'$ that intersect the triangle $\Delta a_1 a_2 a_3$. Observe that any tree or route (not necessarily contained in E) visiting the points in A visits the marked lines as well.

Our algorithm will generate a tree T (not necessarily contained in E) that visits A (and, thus, all marked lines as well) and all the unmarked lines in $\mathcal{L}'$. We will ensure that $|T| \leq \frac{5+4\sqrt{3}}{3} \cdot |T_{\mathrm{opt}}| < 3.98 \cdot |T_{\mathrm{opt}}|$. Consequently, T will be a valid guarding tree for $\mathcal{L}'$ that gives a ratio 3.98 approximation of the optimal solution.

Corners and extremal lines. Consider first the lower right *corner* (vertex) r_1 of E; the other corners of E are handled in a similar way. Consider the set $\mathcal{L}_1' \subset \mathcal{L}'$ of unmarked lines that intersect the triangle $\Delta a_1 r_1 a_2$. Observe that $\Delta a_1 r_1 a_2$ is one of the two corner triangles of $\Delta(a_1 a_2)$, and by

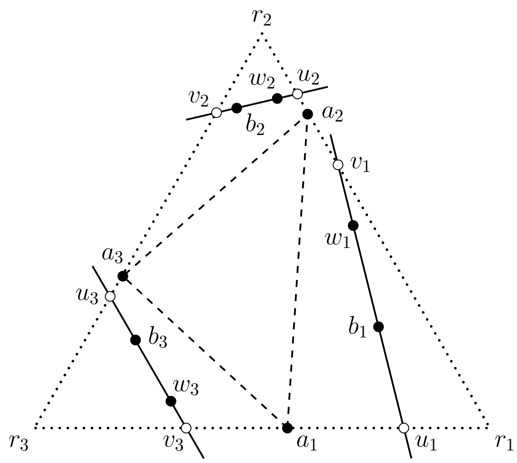

Figure 3: Guessing key elements of an optimal tree. Vertices of the arrangement $\mathcal{A}(\mathcal{L})$ are drawn as filled circles; other points are drawn as empty circles. Lines in $\mathcal{L}'$ are drawn solid.

slightly abusing notation we also refer to $\Delta a_1 r_1 a_2$ as a *corner triangle* of E. If $\mathcal{L}'_1$ is empty, continue with the next corner triangle of E; thus, assume that $\mathcal{L}'_1$ is not empty. Identify an unmarked line $\ell_1 \in \mathcal{L}'_1$ such that the triangle $\Delta u_1 r_1 v_1$ is minimal with respect to inclusion (i.e., no other such triangle is contained in it), where u_1 and v_1 are the intersection points of ℓ_1 with the boundary of E; refer to Fig. 3. The line ℓ_1 is called *extremal.* Note that there may be multiple extremal lines for each corner triangle; the approximation algorithm (described subsequently) will select one such line arbitrarily from those.

Obviously, each line in $\mathcal{L}'_1$ is visited by T_{opt} at some vertex on that line contained in $\Delta a_1 r_1 a_2$. Assume that the extremal line $\ell_1 \in \mathcal{L}'_1$ is visited by T_{opt} at vertex $b_1 \in \ell_1 \cap E$. Let $B = \{b_1, b_2, b_3\}$ be the set of vertices at which T_{opt} visits the extremal lines; we have $0 \leq |B| \leq 3$. The following two technical lemmas are crucial (their proofs are postponed to Section 4.1).

Lemma 1. *The following inequality holds:*

$$|\pi(a_1, b_1)| + |\pi(b_1, a_2)| + |\pi(a_2, b_2)| + |\pi(b_2, a_3)| + |\pi(a_3, b_3)| + |\pi(b_3, a_1)| \leq 2 \cdot |T_{\text{opt}}|.$$

Lemma 2. *Let u_i and v_i be the intersection points of extremal line ℓ_i with the boundary of E, for $i = 1, 2, 3$. The following inequality holds (and is tight in the limit):*

$$|u_1 v_1| + |u_2 v_2| + |u_3 v_3| \leq \frac{2}{\sqrt{3}} \big(|a_1 a_2| + |a_2 a_3| + |a_3 a_1|\big).$$

Approximation algorithm. For each equilateral triangle E determined by (at most) 3 vertices $A \subseteq V(\mathcal{A}(\mathcal{L}))$, with the horizontal lower side, check whether E intersects all lines in $\mathcal{L}'$. If it does not, skip this triangle, since E cannot be the minimal equilateral triangle with a horizontal lower side, containing a minimum-length guarding tree for $\mathcal{L}'$. Otherwise, compute the tree $T = T(E)$ (described below), which intersects all lines. Output the shortest tree among these.

Determine the (at most three) extremal lines ℓ_1, ℓ_2, ℓ_3 corresponding to the three corner triangles of E, as computed in the preprocessing step. That is, for each corner triangle, we arbitrarily select one extremal line from those existent (if any). Consider the line ℓ_1; the same computation is done for the other two lines. Retrieve (from the preprocessing step) the vertex $w_1 \in \ell_1 \cap E$ that minimizes the sum $|\pi(a_1, w_1)| + |\pi(w_1, a_2)|$. Similarly, retrieve $w_i \in \ell_i \cap E$, for $i = 2, 3$. If for some

$i \in \{1,2,3\}$, ℓ_i exists but w_i does not exist, abandon this triangle E, and skip to the next one. (If E is an equilateral triangle containing a minimum-length guarding tree for $\mathcal{L}'$ and ℓ_i exists, then $b_i \in \ell_i \cap E$ exists, hence also $w_i \in \ell_i \cap E$ exists.)

Consider (at most) six paths $\pi_1 = \pi(a_1, w_1)$, $\pi_2 = \pi(w_1, a_2)$, $\pi_3 = \pi(a_2, w_2)$, $\pi_4 = \pi(w_2, a_3)$, $\pi_5 = \pi(a_3, w_3)$, $\pi_6 = \pi(w_3, a_1)$, and assume, without loss of generality, that $|\pi_6| = \max_{i \in \{1,\ldots,6\}} |\pi_i|$. Now, to an initially empty tree T', add all edges of the (at most five) paths $\pi_1, \ldots, \pi_5$. Notice that the temporary graph T' visits $A \cup \{w_1, w_2, w_3\}$ and we do not require that the computed guarding tree T lies inside E. Next, to each vertex w_i, append the edges $u_i w_i$ and $w_i v_i$, for $i = 1, \ldots, 3$ (at most six in total), if not added yet. Output any spanning tree of T' as the final tree T.

Correctness and approximation ratio. First, observe that T is connected, and since it visits A, it intersects all marked lines. Next, each unmarked line intersects a corner triangle $\Delta a_i r_i a_{i+1}$, for some $i = 1,2,3$, where $a_4 = a_1$. Consider an unmarked line ℓ intersecting the triangle $\Delta a_1 r_1 a_2$; the other lines/triangles are handled similarly. Since ℓ_1 is extremal, ℓ either (i) intersects $u_1 v_1$ or (ii) intersects the segment $a_1 u_1$ on the lower side of E and the segment $a_2 v_1$ on the right side of E. In case (i), ℓ intersects one of the edges $v_1 w_1$ and $w_1 u_1$. In case (ii), ℓ separates w_1 from a_1 and from a_2, and, thus, both the paths $\pi_T(a_1, w_1)$ and $\pi_T(a_2, w_1)$, connecting vertices a_1 and w_1, respectively a_2 and w_1, in T, intersect ℓ. Hence in either of the two cases ℓ intersects T. Consequently, T intersects all unmarked lines in $\mathcal{L}'$ as well, and, thus, T is a valid guarding tree for $\mathcal{L}'$.

It remains to show that the weight of the shortest tree over all choices of E, is at most $3.98 \cdot |T_{\text{opt}}|$. Consider the guarding tree T computed when E is the minimal equilateral triangle, with a horizontal lower side, containing T_{opt}. Since $w_1 \in \ell_1$ minimizes the sum $|\pi(a_1, w_1)| + |\pi(w_1, a_2)|$, we have

$$|\pi(a_1, w_1)| + |\pi(w_1, a_2)| \le |\pi(a_1, b_1)| + |\pi(b_1, a_2)|.$$

By adding the analogous inequalities for all three corners of E and using Lemma 1, we obtain

$$\sum_{i=1}^{6} |\pi_i| \le 2 \cdot |T_{\text{opt}}|.$$

Next, since $|\pi_6| = \max_{i \in \{1,\ldots,6\}} |\pi_i|$, we get

$$\sum_{i=1}^{5} |\pi_i| \le \frac{5}{3} \cdot |T_{\text{opt}}|.$$

By the triangle inequality and Lemma 1 we have

$$\begin{aligned} |a_1 a_2| + |a_2 a_3| + |a_3 a_1| &\le |\pi(a_1, a_2)| + |\pi(a_2, a_3)| + |\pi(a_3, a_1)| \\ &\le |\pi(a_1, b_1)| + |\pi(b_1, a_2)| + |\pi(a_2, b_2)| + |\pi(b_2, a_3)| + |\pi(a_3, b_3)| + |\pi(b_3, a_1)| \\ &\le 2 \cdot |T_{\text{opt}}|. \end{aligned} \tag{1}$$

Now Lemma 2 and (1) yield

$$\begin{aligned} |T| &\le \frac{5}{3} \cdot |T_{\text{opt}}| + \big(|u_1 w_1| + |w_1 v_1| + |u_2 w_2| + |w_2 v_2| + |u_3 w_3| + |w_3 v_3|\big) \\ &= \frac{5}{3} \cdot |T_{\text{opt}}| + \big(|u_1 v_1| + |u_2 v_2| + |u_3 v_3|\big) \\ &\le \frac{5}{3} \cdot |T_{\text{opt}}| + \frac{2}{\sqrt{3}} |\big(|a_1 a_2| + |a_2 a_3| + |a_3 a_1|\big) \\ &\le \frac{5}{3} \cdot |T_{\text{opt}}| + \frac{4}{\sqrt{3}} \cdot |T_{\text{opt}}| = \frac{5 + 4\sqrt{3}}{3} \cdot |T_{\text{opt}}| = 3.976.... \cdot |T_{\text{opt}}|. \end{aligned}$$

Running time. The running time of the algorithm is determined by the number of triples of vertices in $V(\mathcal{A}(\mathcal{L}))$ that constitute the set A (or equivalently the number of generated equilateral triangles E) multiplied by the time spent in handling each of the triples. There are at most $\binom{n}{2}^3 = O(n^6)$ such triples, and constructing the guarding tree $T(E)$ for each triple A (triangle E), if $T(E)$ exists (which can be checked in $O(\log n)$ time using the information gathered during preprocessing), requires $O(n)$ time. Recall, $T(E)$ is a spanning tree of the union of at most five shortest paths, each with $O(n)$ edges, and at most six other edges. However, the crucial observation is that when handling each triangle E, we are primarily interested in the length $|T(E)|$ of a guarding tree $T(E)$, and a single tree that minimizes $|T(E)|$ is explicitly constructed as the output. Consequently, since the length $|T(E)|$ can be computed in $O(1)$ time using the information recorded during preprocessing, all but one triple A' minimizing $|T(E)|$ can be handled in $O(\log n)$ time, and handling the triple A', i.e., constructing the output tree, requires $O(n)$ time. Therefore, the total running time is $O(n^6 \log n)$, and Theorem 3 follows.

4.1 Proof of Lemmas 1 and 2

We need the following noncrossing transformations of routes (for points or lines).

Lemma 3. *Let $X \subseteq V(\mathcal{A}(\mathcal{L}))$ be a subset of vertices, and $\mathcal{R}$ be a (polygonal) route for X in the underlying arrangement. Then there exists a (polygonal) route $\mathcal{R}'$ for X that is noncrossing. Moreover, the two routes consist of the same segments and have the same length. In particular,* $\mathrm{conv}(\mathcal{R}) = \mathrm{conv}(\mathcal{R}')$ *holds.*

Proof. If $\mathcal{R}$ is noncrossing, then there is nothing to prove. Assume that $\mathcal{R}$ crosses itself at some vertex $v \in V(\mathcal{A}(\mathcal{L}))$ (observe that $\mathcal{R}$ can only cross at a vertex in $V(\mathcal{A}(\mathcal{L}))$). The vertex v partitions $\mathcal{R}$ into two closed paths (routes) $\mathcal{R}_1$ and $\mathcal{R}_2$ sharing the common vertex v. Then $\mathcal{R}_1 \cup \mathcal{R}_2^R$ has the same length as $\mathcal{R}$ and one fewer crossings than $\mathcal{R}$. Moreover, the two routes consist of the same segments and have the same length. In particular, their convex hulls are the same. One can repeat this operation as long as the resulting routes are self-crossing, until a noncrossing route $\mathcal{R}'$ is obtained — for an illustration of this procedure, see Fig. 4. □

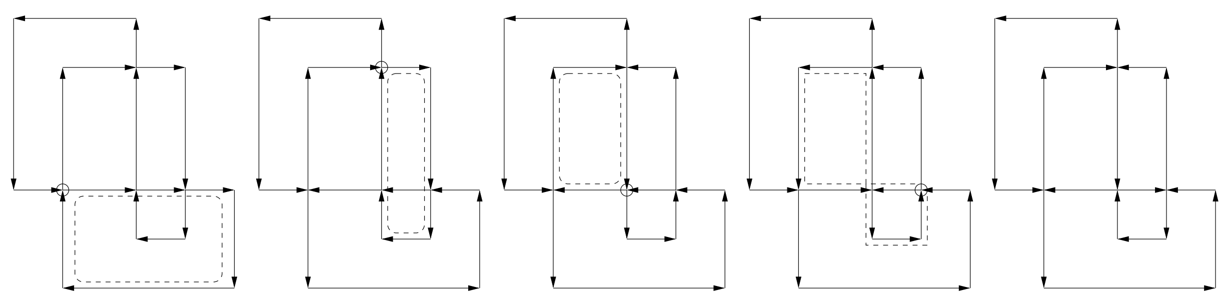

Figure 4: Illustration for the proof of Lemma 3. The route becomes noncrossing in four steps.

The same proof yields:

Lemma 4. *Let $\mathcal{Y} \subseteq \mathcal{L}$ be a subset of lines, and $\mathcal{R}$ be a (polygonal) guarding route for $\mathcal{Y}$ in the underlying arrangement. Then there exists a (polygonal) guarding route $\mathcal{R}'$ for $\mathcal{Y}$ that is noncrossing. Moreover, the two routes consist of the same segments and have the same length. In particular,* $\mathrm{conv}(\mathcal{R}) = \mathrm{conv}(\mathcal{R}')$ *holds.*

Next, we have the following lemma.

Lemma 5. *Let $\mathcal{R}$ be a (polygonal) noncrossing route in $\mathcal{A}(\mathcal{L})$ and let $X \subseteq \mathrm{conv}(\mathcal{R})$ be a subset of vertices of $\mathcal{R}$. Then $\mathcal{R}$ visits the vertices in X in their order along $\mathrm{conv}(\mathcal{R})$.*

Proof. Since $\mathcal{R}$ is noncrossing and closed, assume we move counterclockwise along $\mathcal{R}$. Let $X = \{x_1, x_2, \ldots, x_k\}$, where vertices $x_1, x_2, \ldots, x_k$ appear counterclockwise along $\mathrm{conv}(\mathcal{R})$; without loss of generality assume $|X| \geq 4$.

Consider two vertices $x_1, x_2 \in X$ and suppose that between x_1 and x_2, $\mathcal{R}$ visits some $x_j \in X \setminus \{x_1, x_2\}$. Since $\mathcal{R}$ is noncrossing and $X \subseteq \mathrm{conv}(\mathcal{R})$, after visiting x_2, to make a counterclockwise route, $\mathcal{R}$ must then again visit x_j. Hence we can assume that x_j is visited after visiting x_2. (Notice however that the route remains the same.) By successively applying this reasoning for any vertex in $X \setminus \{x_1, x_2\}$ that appears on $\mathcal{R}$ between vertices x_1 and x_2, we conclude that $\mathcal{R}$ visits all vertices in $X \setminus \{x_1, x_2\}$ only after first visiting x_1 and then x_2.

The same argument can be applied for the other pairs of consecutive vertices in X to complete the proof. □

Now, let E, A and B be defined (depending on $T_{\mathrm{opt}} = T_{\mathrm{opt}}(\mathcal{L}')$) as in the paragraphs "Guessing key elements of an optimal guarding tree" and "Corners and extremal lines". Let P be the convex polygon obtained from E by removing the (at most) three corners cut off by extremal lines ℓ_i, for $i = 1, 2, 3$.

Lemma 6. *There exists a guarding route $\mathcal{R}$ for $A \cup B$ such that*

a) *$\mathcal{R}$ is contained in P;*

b) *$\mathcal{R}$ visits $A \cup B$ in counterclockwise order along $\mathrm{conv}(A \cup B)$, namely, $a_1, b_1, a_2, b_2, a_3, b_3$;*

c) $|\mathcal{R}| \leq 2 \cdot |T_{\mathrm{opt}}|$.

Proof. Start with a shortest guarding route $\mathcal{R}^E_{\mathrm{opt}}$ for $A \cup B$ contained in E. (There exists one since doubling the edges of the guarding tree T_{opt} results in a guarding route for $A \cup B$ contained in E.) By Lemma 4, we can assume that $\mathcal{R}^E_{\mathrm{opt}}$ is noncrossing. Clip $\mathcal{R}^E_{\mathrm{opt}}$ to P along ℓ_1, ℓ_2, and ℓ_3, for each detour of $\mathcal{R}^E_{\mathrm{opt}}$ outside P. Let $\mathcal{R}$ be the resulting route. First, observe that $\mathcal{R}$ is still noncrossing, visits $A \cup B$, and is contained in P; moreover, $|R| \leq |\mathcal{R}^E_{\mathrm{opt}}|$. Next, since $A \cup B \subseteq \mathrm{conv}(\mathcal{R})$, by Lemma 5 we obtain that $\mathcal{R}$ visits $A \cup B$ in counterclockwise order along $\mathrm{conv}(A \cup B)$, namely $a_1, b_1, a_2, b_2, a_3, b_3$. Finally, since doubling the edges of the guarding tree T_{opt} results in a guarding route for $A \cup B$ contained in E, we obtain $|\mathcal{R}^E_{\mathrm{opt}}| \leq 2 \cdot |T_{\mathrm{opt}}|$, and, thus, $|\mathcal{R}| \leq 2 \cdot |T_{\mathrm{opt}}|$, as required. □

Proof of Lemma 1. Consider the route $\mathcal{R}$ from the proof of Lemma 6; recall that $|\mathcal{R}| \leq 2 \cdot |T_{\mathrm{opt}}|$. $\mathcal{R}$ can be partitioned into six paths connecting successive elements of $A \cup B$ in circular counterclockwise order. Since $\pi(\cdot, \cdot)$ are shortest paths in $G(\mathcal{L})$ connecting the same elements in the same order, we have

$$|\pi(a_1, b_1)| + |\pi(b_1, a_2)| + |\pi(a_2, b_2)| + |\pi(b_2, a_3)| + |\pi(a_3, b_3)| + |\pi(b_3, a_1)| \leq |\mathcal{R}|.$$

Putting the two inequalities together yields:

$$|\pi(a_1, b_1)| + |\pi(b_1, a_2)| + |\pi(a_2, b_2)| + |\pi(b_2, a_3)| + |\pi(a_3, b_3)| + |\pi(b_3, a_1)| \leq 2 \cdot |T_{\mathrm{opt}}|.$$

□

Lemma 7. *Let $E = r_1r_2r_3$ be an equilateral triangle, with the horizontal base r_3r_1, and let a_1, a_2 and a_3 be three points such that $a_i \in r_{i-1}r_i$, $i = 1, 2, 3$, with $r_0 = r_3$, see Fig. 5. The following inequality holds (and is tight):*

$$\max\{|a_1r_1|, |r_1a_2|\} + \max\{|a_2r_2|, |r_2a_3|\} + \max\{|a_3r_3|, |r_3a_1|\} \leq \frac{2}{\sqrt{3}} \big(|a_1a_2| + |a_2a_3| + |a_3a_1|\big).$$

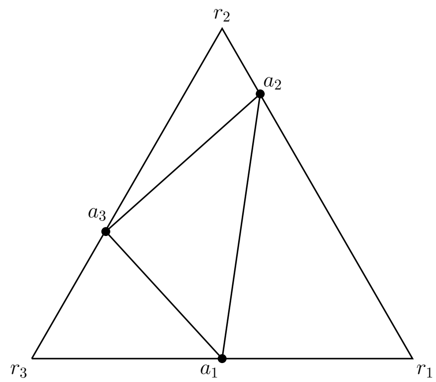

Figure 5: Illustration of Lemma 7.

Proof. Consider the segment a_1a_2; we shall prove that

$$\max\{|a_1r_1|, |r_1a_2|\} \leq \frac{2}{\sqrt{3}} |a_1a_2|.$$

Segments a_2a_3 and a_3a_1 are handled analogously.

Case 1: $\angle a_2a_1r_1 \geq 60^\circ$ (Fig. 6(a)). Consider the triangle $\Delta a_1r_1a_2$, with the base a_1r_1 and the

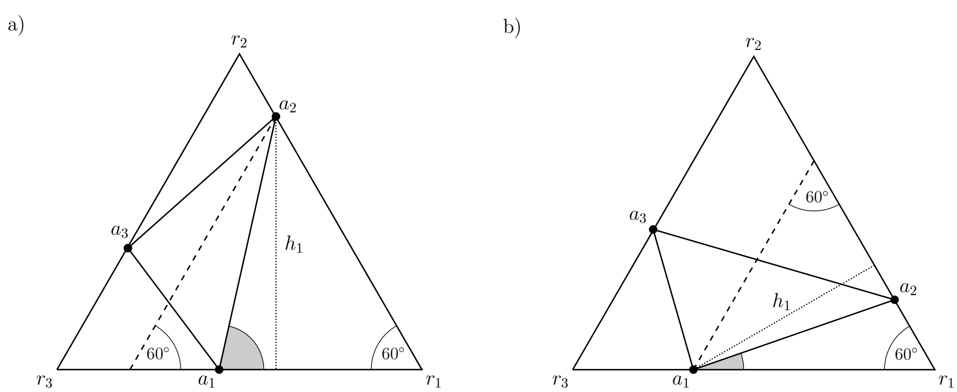

Figure 6: (a) Case 1: $\angle a_2a_1r_1 \geq 60^\circ$. (b) Case 2: $\angle a_2a_1r_1 < 60^\circ$.

height h_1. Since $\angle a_1r_1a_2 = 60^\circ$, we have $h_1 = \frac{\sqrt{3}}{2} |r_1a_2|$, and, thus, since $|a_1a_2| \geq h_1$, we have

$$|r_1a_2| \leq \frac{2}{\sqrt{3}} |a_1a_2|.$$

Finally, since $|r_1a_2| \geq |a_1r_1|$, we obtain

$$|r_1a_2| = \max\{|a_1r_1|, |r_1a_2|\} \leq \frac{2}{\sqrt{3}} |a_1a_2|,$$

as required.

Case 2: $\angle a_2 a_1 r_1 < 60°$ (Fig. 6(b)). Then, $\angle a_1 a_2 r_1 > 60°$, and we may adapt the argument from Case 1. Namely, again consider the triangle $\Delta a_1 r_1 a_2$, now with the base $r_1 a_2$, and the height h_1. Since $\angle a_1 r_1 a_2 = 60°$, we have $h_1 = \frac{\sqrt{3}}{2}|a_1 r_1|$, and, thus, since $|a_1 a_2| \geq h_1$, we have

$$|a_1 r_1| \leq \frac{2}{\sqrt{3}}|a_1 a_2|.$$

Finally, since $|a_1 r_1| > |r_1 a_2|$, we obtain

$$|a_1 r_1| = \max\{|a_1 r_1|, |r_1 a_2|\} \leq \frac{2}{\sqrt{3}}|a_1 a_2|,$$

as required.

The inequality is tight, for instance, when $a_2 = a_3 = r_2$ and a_1 is the midpoint of $r_3 r_1$, in which case both sides are equal to $2|r_1 r_2|$. □

Proof of Lemma 2. Let $u_i v_i$ be a line segment such that $u_i \in a_i r_i$ and $v_i \in r_i a_{i+1}$, for $i = 1, 2, 3$, where $a_4 = a_1$; see Fig. 7. We claim that $|u_i v_i| \leq \max\{|a_i r_i|, |r_i a_{i+1}|\}$, for $i = 1, 2, 3$. It suffices to

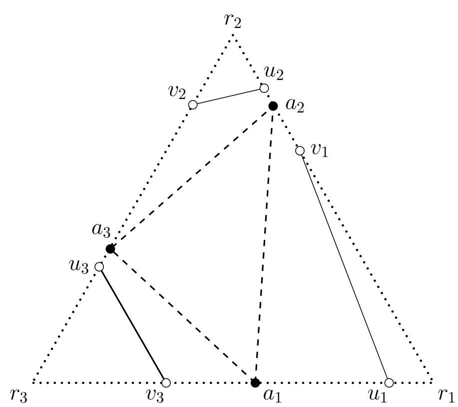

Figure 7: Applying Lemma 7.

prove the claim for $i = 1$; the cases $i = 2$ and $i = 3$ are handled analogously.

Case 1: $\angle r_1 u_1 v_1 \geq 60°$. Then $|u_1 v_1| \leq |r_1 v_1| \leq |r_1 a_2| \leq \max\{|a_1 r_1|, |r_1 a_2|\}$, as required.

Case 2: $\angle r_1 u_1 v_1 < 60°$. Then $|u_1 v_1| \leq |u_1 r_1| \leq |a_1 r_1| \leq \max\{|a_1 r_1|, |r_1 a_2|\}$, as required.

Taking into account the above claim, Lemma 2 immediately follows from Lemma 7.

The inequality in Lemma 2 is tight in the limit, for instance, when a_2 and a_3 are close to r_2, a_1 is the midpoint of $r_3 r_1$, $u_1 v_1 \approx r_1 r_2$, and $u_3 v_3 \approx r_2 r_3$. □

5 Conclusion

In much the same way that we adapted the proof in [9] of the NP-hardness of the shortest watchman route problem for orthogonal line segments to yield a proof of NP-hardness of MGTS (Theorem 1), we can adapt the proof in [9] of the NP-hardness of the shortest watchman route for an arrangement of lines in 3-space to yield a proof of the NP-hardness of the problem of finding a minimum guarding tree for a connected arrangement of lines in 3-space. To this end we use a reduction from the rectilinear Steiner tree problem [11] and obtain the following result.

Theorem 4. *The problem of finding a minimum-length guarding tree for an arrangement of lines in 3-space is NP-hard even for orthogonal lines.*

We conclude with a few open problems.

(i) What is the complexity of MGTL?

(ii) Can the approximation ratios and/or the running times of our algorithms for MGTL be improved?

(iii) What is the complexity of the problem of finding a minimum-length guarding tree for an arrangement of planes in 3-space?

(iv) What is the complexity of the *minimum guarding path* problem for an arrangement of lines in the plane? (Here we are interested in finding the shortest path within the arrangement that visits all lines, or a given subset of lines.)

Acknowledgment

The authors thank Marc Benkert, Étienne Schramm, and Alexander Wolff for valuable remarks and interesting conversations on the topic.

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