Watchman tours for polygons with holes*

Adrian Dumitrescu† Csaba D. Tóth‡

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Abstract

A watchman tour in a polygonal domain (for short, polygon) is a closed curve in the polygon such that every point in the polygon is visible from at least one point of the tour. We show that the length of a minimum watchman tour in a polygon P with k holes is $O(\text{per}(P)+\sqrt{k}\cdot\text{diam}(P))$, where $\text{per}(P)$ and $\text{diam}(P)$ denote the perimeter and the diameter of P, respectively. Apart from the multiplicative constant, this bound is tight in the worst case. We then generalize our result to watchman tours in polyhedra with holes in 3-space. We obtain an upper bound of $O(\text{per}(P)+\sqrt{k\cdot\text{per}(P)\cdot\text{diam}(P)}+k^{2/3}\cdot\text{diam}(P))$, which is again tight in the worst case. Our methods are constructive and lead to efficient algorithms for computing such tours.

We also revisit the NP-hardness proof of the Watchman Tour Problem for polygons with holes.

Keywords: Watchman tours, polygon with holes, approximation algorithm, NP-hardness.

1 Introduction

Visibility and art gallery problems with stationary guards (watchmen) have been studied extensively since the early 1980s [21]. The first question, due to V. Klee was [26]: How many stationary guards are needed to guard the inner walls of any polygon ("art gallery") with n vertices? A follow up question was [17, 26]: How many stationary guards are needed for the surveillance of all the inner walls of a given polygonal domain, and where to position them? If however there is only one guard available, for instance during night time, the guard needs to "tour" the art gallery and make sure that everything is in place. This leads to the next pair of questions: (i) How short can the watchman tour be? (ii) How to find a shortest watchman tour for a given polygon? Previous work on mobile guards [9, 10, 20] focused mostly on the algorithmic aspect, question (ii) above. In this paper we are concerned with estimating the length of a shortest watchman tour, question (i) above.

A *watchman tour* in a polygonal domain (polygon, for short) is a tour (*i. e.*, closed curve) inside the polygon such that every point in the polygon is visible from some point along the tour. Two points in a polygon are visible to each other if the line segment between them lies in the polygon. The watchman tour problem asks for a watchman tour of minimum length [3, 19].

The problem has a polynomial time solution for simple polygons with n vertices (and no holes). Dror *et al.* [13] gave a $O(n^4 \log n)$-time algorithm improving an earlier algorithms by Carlsson *et al.* [7] and respectively Tan [25]. Other variants of the problem are studied in [1, 2, 13, 19, 20].

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†Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: dumitres@uwm.edu. Supported in part by NSF CAREER grant CCF-0444188 and NSF grant DMS-1001667.

‡Department of Mathematics and Statistics, University of Calgary, AB, Canada T2N 1N4. E-mail: cdtoth@ucalgary.ca. Supported in part by NSERC grant RGPIN 35586.

In contrast, computing a shortest watchman tour in a polygon with holes is “known” to be NP-hard [8, 9]. In Section 4, we revisit the old NP-hardness proof by Chin and Ntafos [8, 9] and make some necessary corrections. Specifically, we show that the reduction from Euclidean Traveling Salesman Problem (Euclidean TSP) does not stand, and give a correct reduction from Rectilinear TSP instead. Unfortunately the reference to the incorrect reduction from Euclidean TSP has been perpetuated in the literature over the years; see *e. g.*, [2, p. 203], [19, p. 684], [26, p. 1006].

Our main result is a tight worst-case upper bound for the minimum length of a watchman tour in a polygon with holes. Our upper bound depends on three parameters of a polygon P: the number of holes, $k = k(P)$, the diameter, $\mathrm{diam}(P)$, and the perimeter, $\mathrm{per}(P)$. The diameter of P is the maximum Euclidean distance between any two points in P. The perimeter of P is the total length of the boundary of P, including the boundaries of the holes. In Section 2 we prove the following.

Theorem 1. *The minimum length of a watchman tour for a polygon P with k holes is* $O(\mathrm{per}(P) + \sqrt{k} \cdot \mathrm{diam}(P))$. *This bound cannot be improved for polygons with* $\mathrm{per}(P) > c \cdot \mathrm{diam}(P)$ *for any fixed* $c > 2$. *A watchman tour of this length can be computed in $O(n \log n)$ time, where n is the total number of vertices of P.*

Note that $\mathrm{per}(P) > 2 \cdot \mathrm{diam}(P)$ for every polygon P. If however, $\mathrm{per}(P)$ is very close to $2 \cdot \mathrm{diam}(P)$, then the polygon is long and skinny, and the above upper bound is no longer tight up to constant factors.

Theorem 1 generalizes to 3-dimensional polyhedra, possibly with handles and interior holes. (A handle is a hole penetrating the solid, while an interior hole is an internal void of the solid.) A polyhedron in 3-space is a solid bounded by piecewise linear 2-dimensional manifolds. The 1-skeleton of a polyhedron is the graph consisting of the vertices and edges of the polyhedron. Define the perimeter $\mathrm{per}(P)$ of a polyhedron P as the total length of the edges of P, including the edges of the holes. In Section 3 we prove the following.

Theorem 2. *The minimum length of a watchman tour for a polyhedron P in 3-space with k holes is at most* $O(\mathrm{per}(P) + \sqrt{k \cdot \mathrm{per}(P) \cdot \mathrm{diam}(P)} + k^{2/3} \cdot \mathrm{diam}(P))$. *This bound cannot be improved for polyhedra with* $\mathrm{per}(P) > c \cdot \mathrm{diam}(P)$ *for any fixed* $c > 3$. *A watchman tour of this length can be computed by a randomized algorithm in* $O(k^{4/5} n^{4/5+\delta} + n \log n \log(k + 2))$ *expected time for any* $\delta > 0$*, where n is the total number of vertices, edges, and faces of P.*

Recently and independently of us, Czyzowicz *et al.* [12] designed an exploration algorithm for a robot operating inside a polygon with holes, which produces a robot path whose length is bounded by the same expression $O(\mathrm{per}(P) + \sqrt{k} \cdot \mathrm{diam}(P))$, as in our Theorem 1. The path produced by the exploration algorithm is a watchman path in the polygonal domain, *i. e.*, any point in the domain is visible from some point on the path. The authors did not offer any 3D generalization. Their planar result is stronger in the sense that prior knowledge of the three parameters $k(P)$, $\mathrm{diam}(P)$, and $\mathrm{per}(P)$ is not needed by the algorithm. On the other hand, our algorithm, which is based on different ideas, generalizes to 3-space.

Our results give a partial answer to a question of Nilsson [20], posed in his PhD thesis: “Is it possible to find approximative solutions to guarding problems, with good worst case bounds?”

2 The planar case

In this section we prove Theorem 1. By a classical result of Few [14], the (Euclidean) length of a shortest path through k points in the unit square $[0, 1]^2$ is at most $\sqrt{2k} + 7/4$. Few also proved that the length of a minimum spanning tree of k points in the unit square is at most $\sqrt{k} + 7/4$. Both upper bounds are constructive. The same proofs yield that the corresponding bounds for a square of side length s (instead of 1) are $s\sqrt{2k} + 7s/4$ and $s\sqrt{k} + 7s/4$, respectively.

Few's construction of a short spanning path works as follows. Lay out about $\sqrt{k}$ equidistant horizontal lines, and then visit the points layer by layer, with the path alternating directions along the horizontal strips. An upper bound with a slightly better multiplicative constant for a path has been derived by Karloff [18].

The current best lower bound for the length of such a path is also due to Few: it is $\left(\frac{4}{3}\right)^{1/4} \sqrt{k} - o(\sqrt{k})$, where $(4/3)^{1/4} = 1.075\ldots$. In a related problem, Chung and Graham [11] showed that the length of the shortest *Steiner* tree through k points in the unit square is at most $0.995\sqrt{k}$ (they gave details for an improvement to $0.99995\sqrt{k}$ only). In every dimension $d \geq 3$, Few showed that the maximum length of a shortest path through k points in the unit cube is $\Theta(k^{1-1/d})$.

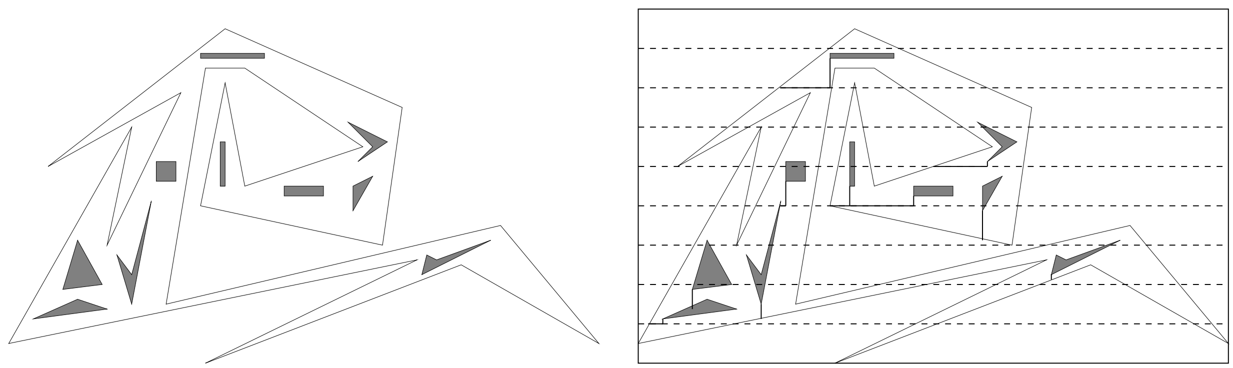

Figure 1: Left: a polygon with $k = 10$ holes (drawn shaded). Right: Connecting the circuits ∂H_i, $i = 0, \ldots, k$ by adding connectors.

Algorithm for constructing a watchman tour. Observe that for a simple polygon P without holes, the boundary ∂P is a watchman tour of length $\text{per}(P)$. Let P be a polygon with $k \geq 1$ holes, $H_1, \ldots, H_k$. For convenience, denote by H_0 the polygon P without the holes. The boundary of P consists of $k+1$ pairwise disjoint circuits $\partial P = \bigcup_{i=0}^{k} \partial H_i$. We have $\text{per}(P) = \sum_{i=0}^{k} |\partial H_i|$, where the vertical bars $|\cdot|$ stand for the Euclidean length.

Our algorithm works as follows. Compute the smallest axis-aligned bounding box B of the polygon P. Then both side lengths of B are at most $\text{diam}(P)$. We augment the disjoint union of circuits ∂H_i, $i = 0, \ldots, k$, with at most $2k$ line segments and possibly (at most $3k$) new vertices to form a connected graph G. The new segments added are called *connectors*. The connectors are either vertical or horizontal. We replace each connector by double edges and obtain a connected multi-graph G' where every vertex has even degree. The watchman tour we construct, W, is an arbitrary Eulerian tour in G', which traverses the entire boundary of P (including the hole boundaries) once, and traverses every connector twice. It is easy to verify that each point p in the interior of P is seen from some point along the tour W: Take an arbitrary line through p and consider its first intersection with the boundary ∂P of P. Since W traverses ∂P, p is visible from some point on $\partial P \subset W$.

It remains to construct the connectors and to bound their total length. As in Few's method, subdivide the bounding box B into horizontal strips by a *raster* of at most $\sqrt{k}$ equidistant horizontal lines such that consecutive raster lines are at $\text{diam}(P)/\sqrt{k}$ distance apart. We construct the connectors in two phases (refer to Fig. 1). Let w_i be a lowest point of H_i, $i = 1, \ldots, k$. In the first phase, for each $i \geq 1$, drop a vertical ray ℓ_i from w_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal raster line. Let v_i denote this vertical segment, and let p_i be its lower endpoint. If point p_i is in the interior of P, then it lies on one of the raster lines. In the second phase, from every point p_i lying in the interior of P, draw a horizontal ray leftwards until it hits the outer boundary, the boundary of another hole, or another point p_j, $j \neq i$. Let h_i denote this horizontal segment. If p_i is already on the boundary of another hole or on ∂H_0, then h_i is a segment of zero length (*i. e.*, not needed).

It is clear that by adding at most $2k$ (horizontal and vertical) connectors $h_i \cup v_i$, we obtain a connected graph G containing all the circuits ∂H_i, $i = 0, \ldots, k$. Indeed, assume without loss of generality that $H_0, H_1, \ldots, H_k$ are listed in increasing order of the y-coordinates of their lowest points. Then each hole H_i, $i \geq 1$, will be connected to some H_j, with $j < i$.

Upper bound. The total length of the horizontal raster lines (inside B) is at most $\text{diam}(P) \cdot \sqrt{k}$, so the total length of the (at most k) horizontal connectors does not exceed this bound. There are k vertical connectors, each of length at most $\text{diam}(P)/\sqrt{k}$. Hence their total length is bounded by $k \cdot \text{diam}(P)/\sqrt{k} = \text{diam}(P) \cdot \sqrt{k}$. Consequently, the total length of W is

$$|W| = |\partial H_0| + \sum_{i=1}^{k} (|\partial H_i| + 2|v_i| + 2|h_i|) = O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P)). \tag{1}$$

Algorithm description and analysis. Let n denote the total number of vertices of P. The bounding box B and the raster lines can be computed in $O(n)$ time. (We do not compute the full arrangement of the raster lines and P, which may have up to $\Theta(n\sqrt{k})$ vertices.) The set of connectors, henceforth the graph G can be computed by a standard line-sweep algorithm [4, 6] in $O(n \log n)$ time. Sweep a horizontal line ℓ top-down. For every position of ℓ, we maintain in sorted order its points of intersection with the vertices and edges of P and with the vertical connectors v_i. This order changes only if ℓ passes through a vertex of P or a point p_i, or if ℓ coincides with a raster line. So there are at most $n + k + \sqrt{k} \leq 3n$ events overall. When the sweep line ℓ coincides with one of the raster lines, we can find the closest intersection point in $\ell \cap \partial P$ to the left of each $p_i \in \ell$ in $O(\log n)$ time.

Observe that the graph G, as well as the multi-graph G' have $O(n)$ edges each. Once G' is constructed, computing an Eulerian tour of G' takes $O(n)$ time. Hence the total time taken by the algorithm is $O(n \log n) + O(n) = O(n \log n)$.

Connectors for arbitrary points in a polygon. Our algorithm for constructing connectors from the points w_i, $i = 1, \ldots, k$, can be applied to an arbitrary set of k points in P. Each point will be connected to some point on the boundary of P, but the holes of P are not necessarily connected to the outer boundary of P. Our bound for the total length of the connectors, however, carries over with the same proof. We thus obtain Lemma 1 below, which is needed later in Section 3.

Lemma 1. *Let P be a polygon (with possible holes), and let S be a set of k points in P. Then there is a forest of total length $O(\sqrt{k} \cdot \text{diam}(P))$ that contains a path from every point in S to some point on the boundary of P. A forest of this length can be computed in $O((n + k) \log(n + k))$ time, where n is the number of vertices of P.*

Lower bound. We now show that our upper bound for the tour length in (1) is tight in the worst case for every $k \geq 0$ and $\mathrm{per}(P) > c \cdot \mathrm{diam}(P)$, where $c > 2$ is a fixed constant. We may assume without loss of generality that $\mathrm{diam}(P) = 1$. We construct a polygon lying in a disk D of unit diameter. If $\mathrm{per}(P) > 2\sqrt{2}$, then let the outer boundary H_0 of P be a square inscribed in D extended with a long and narrow zig-zag "snake" of very small width $0 < \varepsilon \ll 1$ and total edge length $\mathrm{per}(P) - 2\sqrt{2}$ (Fig. 2). The snake lies in D such that the diameter of H_0 is 1. If $c < \mathrm{per}(P) \leq 2\sqrt{2}$, then let H_0 be a rhombus of diameter 1 and side length $(\mathrm{per}(P) - \varepsilon)/4$ for a small $0 < \varepsilon \ll c - 2$. In both cases, we have $\mathrm{per}(H_0) = \mathrm{per}(P) - \varepsilon$, and H_0 contains a square of side length $\Omega(1)$. Arrange k small holes in a grid-like pattern in a maximal square inscribed in H_0. Each hole has $O(1)$ vertices, ε/k perimeter, and a small hidden "cave" that can be seen only by entering it; see *e. g.*, Fig. 4(right) of Section 4.

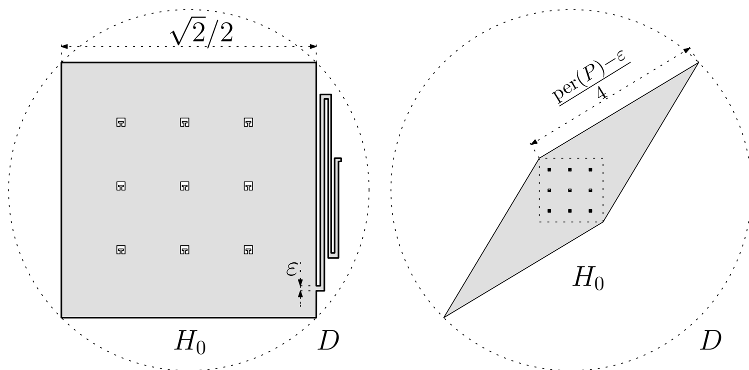

Figure 2: Lower bound constructions for the cases $\mathrm{per}(P) > 2\sqrt{2} \cdot \mathrm{diam}(P)$ and $2 \cdot \mathrm{diam}(P) < \mathrm{per}(P) \leq 2\sqrt{2} \cdot \mathrm{diam}(P)$.

Since the distance between any two holes is $\Omega(1/\sqrt{k})$, it follows that the length of the shortest watchman tour that visits the caves in all holes is $\Omega(k/\sqrt{k}) = \Omega(\sqrt{k})$. If $\mathrm{per}(P) > 2\sqrt{2}$, the length of any walk from the bottom of the zig-zag snake to one of the furthest caves is $\Omega(\mathrm{per}(P))$. We conclude that in both cases the length of the shortest watchman tour for P is

$$\Omega(\mathrm{per}(P) + \sqrt{k}) = \Omega(\mathrm{per}(P) + \sqrt{k} \cdot \mathrm{diam}(P)),$$

as required. This completes the proof of Theorem 1.

Remark. The bound in Theorem 1 depends on c only if c is close to 2. If $c = 2 + \lambda$, and $0 < \lambda \leq 1$, say, then the two axes of the rhombus in the lower bound construction are $\mathrm{diam}(P)$ and $\Theta(\sqrt{\lambda} \cdot \mathrm{diam}(P))$. By taking all lattice points of a fine grid instead of a square section of the lattice inside the rhombus, the resulting lower bound is $\Omega(\mathrm{per}(P) + \lambda^{1/4}k^{1/2} \cdot \mathrm{diam}(P))$, which becomes $\Omega(\mathrm{per}(P))$ when λ tends to zero.

3 Generalization to 3-dimensions

In this section we prove Theorem 2. Observe that every point p in the interior of P sees at least one point on some edge of P. Indeed, consider a plane h containing p that is not incident to any vertex of P. The intersection $P \cap h$ is a *planar map* made up from disjoint polygonal cycles; from any two adjacent regions of the map, one is exterior and the other is interior to P. Refer to Fig. 3 for an example. The point p lies in an interior region of the map. In a triangulation of this region, p lies in a triangle, and hence it sees the three vertices of the triangle. All three vertices are intersection points of h with some edges of P. It follows that a tour that traverses every edge of P is a watchman tour: *i. e.*, every interior point of P is seen from some point of the tour.

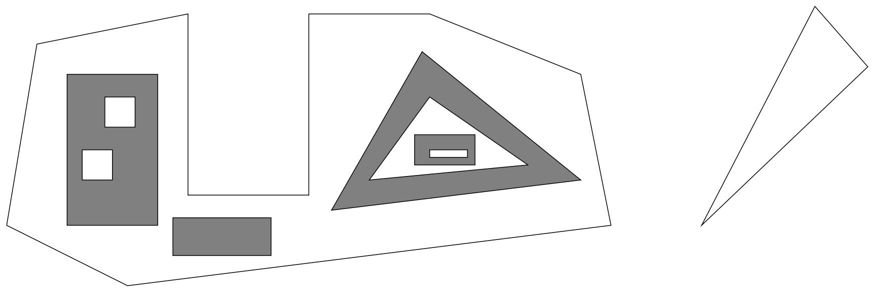

Figure 3: The intersection of the polyhedron P with a plane π forms a planar map. The interior holes of P and the exterior regions enclosed by the the intersection $\pi \cap \partial P$ are shaded.

Algorithm, its analysis and upper bound. Our algorithm for computing a watchman tour for P is analogous to the planar case. First augment the 1-skeleton of P to obtain a connected graph G. Then double some of the edges of G to make all vertex degrees even. The watchman tour we generate is an arbitrary Eulerian tour in this multi-graph.

The input is a polyhedron P with $k \geq 1$ holes $H_1, \ldots, H_k$. Denote by H_0 the outer boundary of P. Let w_i be a lowest point of H_i, $i = 1, \ldots, k$. Assume without loss of generality that $H_1, \ldots, H_k$ are listed in increasing order of the z-coordinates of their lowest points. Compute an axis-aligned bounding box B of the polyhedron P of side length at most $\mathrm{diam}(P)$. Subdivide B into horizontal strips by a *raster* of at most $k^{1/3}$ equidistant horizontal planes such that consecutive raster planes are at $\mathrm{diam}(P)/k^{1/3}$ distance apart. Subdivide every strip by additional horizontal planes, if necessary, such that there are at most $k^{2/3}$ points w_i between consecutive horizontal planes. We have used at most $2k^{1/3}$ horizontal planes. From each w_i, drop a vertical ray ℓ_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal plane. Let p_i be the lower endpoint of this vertical segment.

If point p_i is in the interior of P, then it lies on some horizontal plane. In each horizontal plane, we invoke a modified version of our planar algorithm with $k' = k^{2/3}$, for constructing connectors from every point p_i to the outer boundary or the boundary of another hole. While the planar algorithm was designed to work on a polygon with holes, it can be adapted to work on the planar map in the corresponding horizontal plane. The total length of the vertical and horizontal connectors in the $O(k^{1/3})$ planes is bounded by

$$O\left(k \cdot (\mathrm{diam}(P)/k^{1/3}) + k^{1/3} \cdot (\sqrt{k^{2/3}} \cdot \mathrm{diam}(P))\right) = O\left(k^{2/3} \cdot \mathrm{diam}(P)\right). \qquad (2)$$

For each hole H_i, $i \geq 1$, we have computed a connector from a lowest vertex w_i to some point on the boundary of H_j, for some $j < i$. However, the endpoint of a connector may lie in the interior of a face of P. For every face f of P, let k_f denote the number of connector endpoints in the interior of f, with $\sum_f k_f \leq k$. In each face f, with $k_f \geq 1$, we apply Lemma 1 to construct a forest that connects these k_f points to points on the boundary of f. The length of each forest is $O(\sqrt{k_f} \cdot \mathrm{diam}(f))$.

Observe the following inequalities, where f is any face of P:

$$\max_f \mathrm{diam}(f) \leq \mathrm{diam}(P) \quad \text{and} \quad \mathrm{diam}(f) \leq \frac{1}{2}\mathrm{per}(f).$$

Since every edge is adjacent to exactly two faces, we also have

$$\sum_f \mathrm{diam}(f) \le \sum_f \frac{1}{2}\mathrm{per}(f) \le \mathrm{per}(P).$$

Applying the Cauchy-Schwarz inequality yields that the total length of these spanning trees is bounded from above as follows:

$$\begin{aligned} & O\left(\sum_f \sqrt{k_f}\cdot \mathrm{diam}(f)\right) = O\left(\sqrt{\sum_f k_f}\sqrt{\sum_f \mathrm{diam}^2(f)}\right) \\ = \; & O\left(\sqrt{k}\sqrt{\mathrm{diam}(P)}\sqrt{\sum_f \mathrm{diam}(f)}\right) = O\left(\sqrt{k\cdot \mathrm{diam}(P)\cdot \mathrm{per}(P)}\right). \end{aligned} \tag{3}$$

By adding the term $\mathrm{per}(P)$ to the lengths in (2) and (3) the upper bound on the length of the tour in Theorem 2 follows.

Since the total number of vertices, edges and faces of P is n, the faces of P can be triangulated into $O(n)$ triangles in $O(n)$ time [6]. All vertical connectors can be computed by a (randomized) batched ray shooting algorithm due to Pellegrini [23, Theorem 4]. Given $O(n)$ interior-disjoint triangles in 3-space, k batched vertical ray shooting queries take $O(k^{4/5}n^{4/5+\delta} + n\log n\log(k+2))$ expected time for any $\delta > 0$, where the constant of proportionality depends on δ. Similarly, all horizontal connectors can be computed in $O(k^{4/5}n^{4/5+\delta} + n\log n\log(k+2))$ expected time. All connector forests for the $O(k)$ connector endpoints in the faces of P can be computed in $\sum_f O(n_f + k_f)\log(n_f + k_f)) = O(n\log n)$ time by Lemma 1. If P has no holes ($k = 0$), neither triangulation nor ray shooting is necessary, hence the algorithm is deterministic and runs in $O(n)$ time.

Alternatively, using the data structure of de Berg [5, 24] for online ray shooting in a fixed direction, the $O(k)$ connectors can be computed deterministically in $O(kn^{2/3+\delta} + n^{1+\delta})$ time for any $\delta > 0$, where the constant of proportionality depends on δ.

Lower bound. The lower bound constructions are similar to the planar case. We construct a polyhedron P with k holes and $\mathrm{per}(P) > c\cdot\mathrm{diam}(P)$ for a fixed $c > 3$. Note that $\mathrm{per}(P) > 3\cdot\mathrm{diam}(P)$ for every polyhedron, since the 1-skeleton of the outer boundary of P is a 3-connected graph, hence it contains at least three edge-disjoint paths between any two vertices. In particular, there are at least three edge-disjoint paths between two vertices at $\mathrm{diam}(P)$ distance apart, and at most one of these paths may have length exactly $\mathrm{diam}(P)$. Hence $\mathrm{per}(P) > 3\cdot\mathrm{diam}(P)$.

We may assume without loss of generality that $\mathrm{diam}(P) = \Theta(1)$. We present three lower bound constructions, one for each term in the upper bound $O(\mathrm{per}(P) + \sqrt{k\cdot\mathrm{per}(P)} + k^{2/3})$.

Case 1: $c < \mathrm{per}(P) \le k^{1/3}$. In this case, $\max(\mathrm{per}(P), \sqrt{k\cdot\mathrm{per}(P)}, k^{2/3}) = k^{2/3}$. Let H_0 be a double pyramid obtained by gluing together two congruent pyramids, each with $1/2$ height, and with a (small) triangular base. Arrange the holes in a grid-like pattern in a maximal inscribed cube of side length $\Omega(1)$ in H_0, each with a tiny hidden "cave." By Few's result [14], a watchman tour that visits every cave has length $\Omega(k^{2/3})$.

Case 2: $k^{1/3} < \mathrm{per}(P) \le k$. In this case, $\max(\mathrm{per}(P), \sqrt{k\cdot\mathrm{per}(P)}, k^{2/3}) = \sqrt{k\cdot\mathrm{per}(P)}$. Let H_0 be composed of about $\Theta(\mathrm{per}(P))$ homothetic copies of a flat box of size $1\times 1\times\varepsilon$, for some small $\varepsilon > 0$, connected by a narrow corridor. Arrange about $\Theta(k/\mathrm{per}(P))$ holes, each with a hidden "cave," in a planar grid-like pattern in each flat box. As in the planar case, the length of the watchman tour within each flat box is $\Omega(\sqrt{k/\mathrm{per}(P)})$, and so the total length is $\Omega(\sqrt{k\cdot\mathrm{per}(P)})$.

Case 3: $k < \text{per}(P)$. In this case, $\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3}) = \text{per}(P)$. Let H_0 be a narrow zig-zag "snake" of diameter 1, built of a sequence of triangular prisms (and arbitrary holes of very small perimeter). A tour that visits both ends of the snake must have $\Omega(\text{per}(P))$ length.

In all three cases, we have shown that there is a polyhedron P with k holes, $\text{per}(P)$ perimeter and $\Theta(1)$ diameter such that the length of every watchman tour is $\Omega(\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3})) = \Omega(\text{per}(P) + \sqrt{k \cdot \text{per}(P)} + k^{2/3})$, as required. This completes the proof of Theorem 2.

Remark. Similar to the planar case, the bound in Theorem 2 depends on c only if c is close to 3. If $c = 3 + \lambda$, and $0 < \lambda \leq 3$, say, by taking all lattice points of a fine grid instead of a cubic section of the lattice inside a double pyramid of diameter $\text{diam}(P)$, the resulting lower bound is $\Omega(\text{per}(P) + \lambda^{1/3} k^{2/3} \cdot \text{diam}(P))$, which becomes $\Omega(\text{per}(P))$ when λ tends to zero.

4 NP-hardness proof revisited

In this section we revisit the old NP-hardness proof of the watchman tour problem by Chin and Ntafos [8, 9]. We show that the reduction from Euclidean TSP (ETSP) used in [8, 9] does not stand, and give a correct reduction from Rectilinear TSP (RTSP) instead.

Recall that the Euclidean (or L_2) distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

while the rectilinear (Manhattan, or L_1) distance between these points is $|x_1 - x_2| + |y_1 - y_2|$. It is known [15, 22] that both ETSP and RTSP are NP-hard. The Watchman Tour Problem (WTP) and the two geometric variants of TSP (ETSP and RTSP) [16, 22] can be formulated as decision problems as follows:

WTP: Given a polygon P with k polygonal holes, and a positive integer m, does there exist a watchman tour of total Euclidean length at most m?

ETSP: Given a set of n points in the plane, and a positive integer m, does there exist a tour of total Euclidean length at most m that visits all the points?

RTSP: Given a set of n points in the plane, and a positive integer m, does there exist a tour of total rectilinear length at most m that visits all the points?

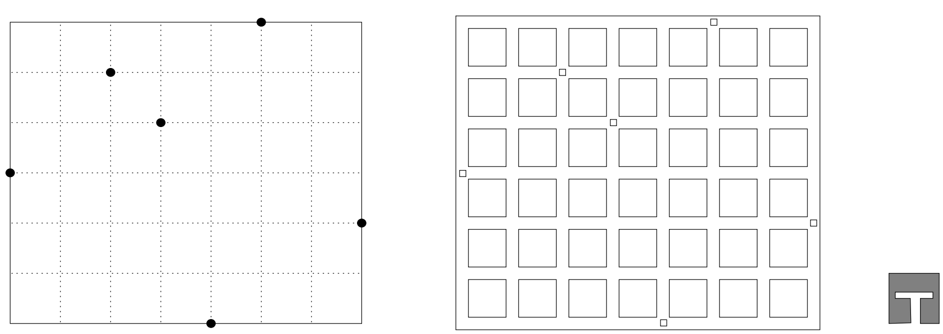

Figure 4: Left: point set S. Middle: polygon P with holes. Right: a small hole for each point in S.

The NP-hardness proof for WTP in [8] and similarly that in [9] use an incorrect reduction from the Euclidean TSP to the Watchman Tour Problem via a claim that relates the length of a solution for ETSP to the length of a solution for WTP: [8, Theorem 1, p. 25] and [9, Theorem 2.1, p. 40]. The claimed relation of the form $\mathrm{OPT}(\mathrm{WTP}) = \mathrm{OPT}(\mathrm{ETSP}) + \Delta$ is incorrect, as no such equality holds in their construction for a fixed Δ. Corollary 1 in [8] and Corollary 2.2 in [9] add even more to the confusion by mentioning RTSP in regard to a side issue of rectilinear polygons. Here we give a correct NP-hardness reduction from the Rectilinear TSP to the Watchman Tour Problem.

Given a set S of n lattice points, assume without loss of generality that the smallest axis-aligned rectangle containing S is $R = [0, a] \times [0, b]$, for some positive integers $b \leq a$; see Fig. 4. Construct a polygon P with $ab+n$ holes as follows. The outer boundary of P is obtained a slightly enlarging R around its boundary. We have ab large holes formed by the cells of the grid, but only slightly smaller so that they are disjoint. We also have n small holes corresponding to the points in S, each containing a "cave" that can only be seen by entering it. The midpoint of the top wall of the cave is the corresponding point in S; we call it the *reference* point of the hole.

To be precise, the width of the narrow corridors in P left by the big holes is set to $w = 1/(6an)$. The small holes are small enough so that they fit in the narrow corridors left by the big grid cell holes: the perimeter of each small hole is set to $p = \frac{w}{2} = 1/(12an)$. The outer boundary of P is $H_0 = [-w/2, a+w/2] \times [-\frac{w}{2}, b+\frac{w}{2}]$, while the ab large holes are $[i+\frac{w}{2}, i+1-\frac{w}{2}] \times [j+\frac{w}{2}, j+1-\frac{w}{2}]$ for $0 \leq i \leq a-1$ and $0 \leq j \leq b-1$. Between any two adjacent large holes, there is a narrow rectangular corridor of side lengths $1-w$ and w. There are also narrow corridors of the same dimensions between the outer boundary and the adjacent large holes.

The reduction, hence the NP-hardness, follows via the following claim.

Claim. For a positive integer m, there exists a tour of S of rectilinear length m if and only if there exists a watchman tour of P of Euclidean length $m + \delta$, with $-1/3 \leq \delta \leq 1/3$.

To verify the claim observe first that the rectilinear distance between any two (lattice) points in S is an integer. Hence the rectilinear length of the shortest tour of the points in S is an integer, say m. Observe also that any tour of the points can be converted into a tour of the polygon, and viceversa, by visiting the points in S (or the caves in the small holes) in the same order. Moreover, as we show below, the lengths of the two tours are very close to each other. Indeed, on one hand, by making only small detours from any given TSP tour for S yields a WTP tour of P. On the other hand, a WTP tour for P can be converted into a TSP tour for S whose length is very close to the original one. Note the following two properties of sub-paths connecting points in two consecutive caves in a WTP tour:

1. If the L_1-distance between two points in S is an integer d, $1 \leq d \leq a+b$, then any path in P between two interior points in two corresponding caves has length at least $d(1-w)$, since any path has to traverse at least d narrow corridors.
2. If the L_1-distance between two points in S is an integer d, then there is a path in P of length at most $d+(d+1)p$ between any two interior points in the corresponding caves, since it takes a detour no longer than $2p = w$ to go into the cave and around a small hole along the way. Note however that when converting an optimal RTSP tour for S into a WTP tour for P, at most n such small detours are needed, and each adds at most w to the total length.

It follows that the rectilinear length of the shortest tour of the n points in S can differ from the (Euclidean) length of the shortest watchman tour of P by at most $n(a+b)w = n(a+b)/(6an) \leq 1/3$, as required.

Observe that the integrality requirement for m is crucial. Moreover, no such claim holds if the length of the tour of the points in S is measured in the L_2 metric, or equivalently, Euclidean TSP is used in the reduction.

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