On stars and Steiner stars *

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Abstract

A *Steiner star* for a set P of n points in $\mathbb{R}^d$ connects an arbitrary point in $\mathbb{R}^d$ to all points of P, while a *star* connects one of the points in P to the remaining $n-1$ points of P. All connections are realized by straight line segments. Let the *length* of a graph be the total Euclidean length of its edges. Fekete and Meijer showed that the minimum star is at most $\sqrt{2}$ times longer than the minimum Steiner star for any finite point configuration in $\mathbb{R}^d$. The supremum of the ratio between the two lengths, over all finite point configurations in $\mathbb{R}^d$, is called the *star Steiner ratio* in $\mathbb{R}^d$. It is conjectured that this ratio is $4/\pi = 1.2732\ldots$ in the plane and $4/3 = 1.3333\ldots$ in three dimensions. Here we give upper bounds of 1.3631 in the plane, and 1.3833 in 3-space. These estimates yield improved upper bounds on the maximum ratios between the minimum star and the maximum matching in two and three dimensions. We also verify that the conjectured bound $4/\pi$ in the plane holds in two special cases.

Our method exploits the connection with the classical problem of estimating the maximum sum of pairwise distances among n points on the unit sphere, first studied by László Fejes Tóth. It is quite general and yields the first non-trivial estimates below $\sqrt{2}$ on the star Steiner ratios in arbitrary dimensions. We show, however, that the star Steiner ratio in $\mathbb{R}^d$ tends to $\sqrt{2}$ as d goes to infinity. As it turns out, our estimates are related to the classical infinite Wallis product: $\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2-1}\right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$.

1 Introduction

Let P be a set of n points in $\mathbb{R}^d$. A *Steiner star* for P connects an arbitrary point in $\mathbb{R}^d$ to all points of P, while a *star* connects one of the points in P to the remaining $n-1$ points of P. All connections are realized by straight line segments. The study of minimum Steiner stars and minimum stars is motivated by applications in facility location and computational statistics [7, 8, 9, 14]. The *Weber point* (or Weber center), also known as the *Fermat-Torricelli point* or *Euclidean median*, is the point of the space that minimizes the sum of distances to n given points in $\mathbb{R}^d$. It is known that even in the plane, the Weber point cannot be computed exactly, already for $n \geq 5$ [5, 10]. For $n = 3$ and 4, resp., Torricelli and Fagnano gave algebraic solutions. The Weber point can however be approximated with arbitrary precision [8, 9], based on Weiszfeld's algorithm [19]. The problem of finding such a point can be asked in any metric space. The Fermat-Torricelli problem has been also studied in Minkowski spaces [18]. More information on the problem can be found in [11].

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Ratio	Lower bound	Old upper bound	New upper bound
$\rho_2 : L(S_{\min})/L(SS_{\min})$	$\frac{4}{\pi} = 1.2732\ldots$	$\sqrt{2} = 1.4142\ldots$	1.3631 †
$\rho_3 : L(S_{\min})/L(SS_{\min})$	$\frac{4}{3} = 1.3333\ldots$	$\sqrt{2} = 1.4142\ldots$	1.3833 †
$\rho_4 : L(S_{\min})/L(SS_{\min})$	$\frac{64}{15\pi} = 1.3581\ldots$ †	$\sqrt{2} = 1.4142\ldots$	1.3923 †
$\rho_5 : L(S_{\min})/L(SS_{\min})$	$\frac{48}{35} = 1.3714\ldots$ †	$\sqrt{2} = 1.4142\ldots$	1.3973 †
$\rho_{100} : L(S_{\min})/L(SS_{\min})$	$1.4124\ldots$ †	$\sqrt{2} = 1.4142\ldots$	1.4135 †
$\eta_2 : L(S_{\min})/L(M_{\max})$	$\frac{4}{3} = 1.3333\ldots$	1.6165	1.5739 †
$\eta_3 : L(S_{\min})/L(M_{\max})$	$\frac{3}{2} = 1.5$	1.9999	1.9562 †

Table 1: Lower and upper bounds on star Steiner ratios $\rho_2, \rho_3, \rho_4, \rho_5, \rho_{100}$, and matching ratios η_2, η_3 for some small values of d. Those marked with † are new.

We now introduce some definitions and notations, following [16]. Consider a finite point set P in $\mathbb{R}^d$. Let $S_{\min}$ denote a minimum star, $SS_{\min}$ denote a minimum Steiner star, and $M_{\max}$ denote a maximum matching (for an even number of points). Let $\rho_d(P)$ denote the maximum ratio between the lengths of the minimum star and the minimum Steiner star for P. Let $\rho_d(n)$ be the supremum of $\rho_d(P)$ over all point sets of size n in $\mathbb{R}^d$. Finally, let ρ_d be the supremum of $\rho_d(n)$ over all positive integers n. Obviously $\rho_d(n) \leq \rho_d$, for each n. Fekete and Meijer [16] were the first to study the star Steiner ratio. They proved that $\rho_d \leq \sqrt{2}$ holds for any dimension d. It is conjectured that $\rho_2 = 4/\pi = 1.2732\ldots$, and $\rho_3 = 4/3 = 1.3333\ldots$, which are the limit ratios for a uniform mass distribution on a circle, and a sphere, respectively (in these cases the Weber point is the center of the circle, or sphere) [16].

For a given finite even-size point set P in $\mathbb{R}^d$, let $\eta_d(P)$ denote the maximum ratio between the lengths of the minimum star and that of a maximum matching of P. Let $\eta_d(n)$ be the supremum of $\eta_d(P)$ over all point sets of size n in $\mathbb{R}^d$ (n even). Finally, let η_d be the supremum of $\eta_d(n)$ over all positive even integers n. Obviously $\eta_d(n) \leq \eta_d$, for each even n. By using their bounds on ρ_d, Fekete and Meijer also established nontrivial bounds on η_d in two and three dimensions ($d = 2, 3$).

In this paper we make further progress and obtain better bounds on the ratios ρ_d and η_d, by exploiting the connection with the classical problem of estimating the maximum sum of pairwise distances among n points on the unit sphere, first studied by László Fejes Tóth [15]. We first prove that $\rho_2 \leq 1.3631$, and $\rho_3 \leq 1.3833$. We also show that the conjectured bound $4/\pi$ in the plane holds in two special cases, corresponding to the lower bound construction. Based on the above estimates on ρ_2 and ρ_3, we then further obtain improved estimates on η_2 and η_3, using the method developed by Fekete and Meijer [16]. Our improvements are summarized in Table 1. Finally, our method yields the first non-trivial estimates below $\sqrt{2}$ on the star Steiner ratios in arbitrary dimensions. Among others, we show that the upper bound $\sqrt{2}$ on the star Steiner ratio given by Fekete and Meijer is in fact a very good approximation for higher dimensions d. Thus in this sense the problem in the plane is the most interesting one, and the gap between the upper and the lower bound on the star Steiner ratio is the largest.

Among n points, $q_0, q_1, \ldots, q_{n-1}$, on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the sum of pairwise Euclidean distances is $\sum_{i<j} |q_i q_j|$. Let $G(d, n)$ denote the maximum of $\sum_{i<j} |q_i q_j|$ over all n-element point sets $\{q_0, q_1, \ldots, q_{n-1}\} \subset \mathbb{S}^{d-1}$. As in [1, 3, 4], define the *constant of uniform density for the sphere* in $\mathbb{R}^d$, c_d, as the average distance from a point of $\mathbb{S}^{d-1}$ to all other points of $\mathbb{S}^{d-1}$. That is,

$$c_d = \frac{\int_{\mathbb{S}^{d-1}} |pq| \, \mathrm{d}q}{\int_{\mathbb{S}^{d-1}} \, \mathrm{d}q}, \quad \text{for } d \geq 2$$

for any point $p \in \mathbb{S}^{d-1}$. For example, $c_1 = 1$, $c_2 = 4/\pi$, and $c_3 = 4/3$ (more details in Section 3). The sequence c_d, $d \geq 1$, is increasing and our Theorem 2 shows that $\lim_{d\to\infty} c_d = \sqrt{2}$. The connection with this problem will be evident in the next section.

Björck and others [6] have shown that

$$G(d,n) \leq \frac{c_d}{2} \cdot n^2 \quad \text{and} \quad \lim_{n\to\infty} \frac{G(d,n)}{n^2} = \frac{c_d}{2}. \tag{1}$$

László Fejes Tóth [15] gave a nice closed formula for $G(2,n)$ for every $n \geq 2$:

$$G(2,n) = \frac{n}{\tan\frac{\pi}{2n}} = \frac{2}{\pi}n^2 - \frac{\pi}{6} + O\left(\frac{1}{n^2}\right). \tag{2}$$

The *exact* determination of $G(d,n)$ for $d \geq 3$ is considered to be a difficult geometric discrepancy problem [3, pp. 298]. For $d = 3$, Alexander [1, 2] has shown that

$$\frac{2}{3}n^2 - 10n^{1/2} < G(3,n) < \frac{2}{3}n^2 - \frac{1}{2}. \tag{3}$$

2 Stars in $\mathbb{R}^d$

A *geometric graph* G in $\mathbb{R}^d$ is a pair (V,E) where V is a finite set of points in $\mathbb{R}^d$, and E is a set of segments (edges) connecting points in V. The *length* of G, denoted $L(G)$, is the sum of the Euclidean lengths of all edges in G.

Let $P = \{p_0, \ldots, p_{n-1}\}$ be a set of n points in $\mathbb{R}^d$. Let $SS_{\min}$ be a minimal Steiner star for P, and assume that its center c is *not* an element of P. As noted in [16], the minimality of the Steiner star implies that the sum of the unit vectors rooted at c and oriented to the points vanishes, that is, $\sum_{i=0}^{n-1} \frac{\overrightarrow{cp_i}}{|\overrightarrow{cp_i}|} = \overrightarrow{0}$. For completeness, we include here the brief argument. Fix an arbitrary orthogonal coordinate system in $\mathbb{R}^d$. Assume that $p_i = (p_{i1}, p_{i2}, \ldots, p_{id})$, for $i = 0, 1, \ldots, n-1$. The length of the star S_x centered at an arbitrary point $x = (x_1, x_2, \ldots, x_d)$ is

$$L(S_x) = \sum_{i=0}^{n-1} |xp_i| = \sum_{i=0}^{n-1} \sqrt{\sum_{j=1}^{d} (x_j - p_{ij})^2}.$$

If $c = (c_1, c_2, \ldots, c_d)$ is the Weber point, then all partial derivatives of $L(S_x)$ vanish at $x = c$:

$$\left.\frac{\partial L(S_x)}{\partial x_j}\right|_{x=c} = \sum_{i=0}^{n-1} \frac{c_j - p_{ij}}{\sqrt{\sum_{j=1}^{d}(c_j - p_{ij})^2}} = 0, \qquad \text{for } j = 1, 2, \ldots, d. \tag{4}$$

Our setup is as follows. Refer to Figure 1 for an example in the plane. We may assume that the Weber point is not in P, since otherwise the star Steiner ratio is 1. Choose the coordinate system such that the Weber point is the origin o and the closest point in P to o is $p_0 = (1, 0, \ldots, 0)$, hence $L(SS_{\min}) = (1+\delta)n$, for some $\delta \geq 0$. For $i = 0, 1, \ldots, n-1$, let q_i be the intersection point of line segment op_i and the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Then $\overrightarrow{oq_i} = \frac{\overrightarrow{op_i}}{|\overrightarrow{op_i}|}$, that is, $\overrightarrow{oq_i}$ is the unit vector corresponding to $\overrightarrow{op_i}$. Let $a_i = |op_i|$, $b_i = |p_0q_i|$, and $a_i' = |p_0p_i|$, for $i = 0, 1, \ldots, n-1$. We have $L(SS_{\min}) = \sum_{i=0}^{n-1} a_i$. Finally, let $\alpha_{ij} \in [0, \pi]$ be the angle between the positive x_j-axis and line segment op_i. Henceforth (4) can be rewritten in the more convenient form

$$\sum_{i=0}^{n-1} \cos\alpha_{ij} = 0, \qquad \text{for } j = 1, 2, \ldots, d. \tag{5}$$

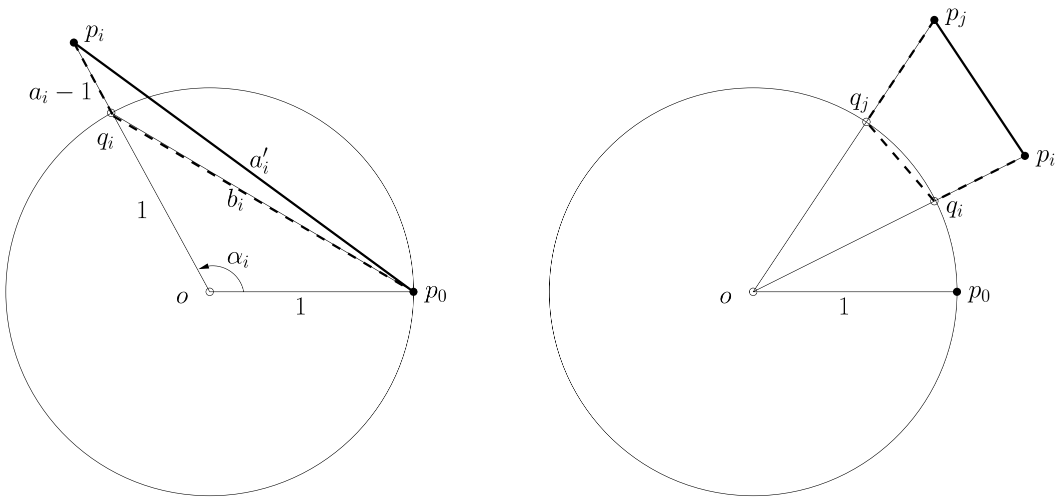

Figure 1: Left: estimating the length of a star centered at $p_0 = (1, 0) \in P$. Right: estimating pairwise distances.

Let S_i be the star centered at p_i, for $i = 0, 1, \ldots, n-1$. By definition, we have

$$L(S_{\min}) = \min_{0 \le i \le n-1} L(S_i).$$

Using the local optimality condition (5), Fekete and Meijer [16] show that if one moves the center of $SS_{\min}$ from the Weber point to a closest point of P, the sum of distances increases by a factor of at most $\sqrt{2}$, and this bound is best possible. It follows (see [16] for details) that for any $d \ge 2$,

$$\rho_d(n) \le \sqrt{2}, \ \text{ thus } \ \rho_d \le \sqrt{2}. \tag{6}$$

From the opposite direction, by considering a uniform mass distribution on the unit sphere in $\mathbb{R}^d$, one has for any $d \ge 2$:

$$c_d \le \rho_d. \tag{7}$$

Our new argument in a nutshell is as follows. If δ is large, we consider the star centered at a point in P closest to the Weber point, as a good candidate for approximating the minimum star. If δ is small, we upper bound the length of the minimum star by the average of all n stars. In the end we balance the two estimates obtained. Applying the averaging argument (for small δ) leads naturally to the problem of maximizing the sum of pairwise distances among n points on the unit sphere (or unit circle).

Theorem 1 *Let $d \ge 2$, and let c_d denote the constant of uniform density for the sphere in $\mathbb{R}^d$. The star Steiner ratio in $\mathbb{R}^d$ is bounded as follows:*

$$c_d \le \rho_d \le \frac{2\sqrt{2} - c_d}{1 + \sqrt{2} - c_d}. \tag{8}$$

Proof. Consider an n-element point set P in $\mathbb{R}^d$, and the setup above. It is enough to show that

$$\frac{L(S_{\min})}{L(SS_{\min})} \le \frac{2\sqrt{2} - c_d}{1 + \sqrt{2} - c_d} \, .$$

By the triangle inequality (see also Fig. 1(left)), we have $a'_i \leq b_i + a_i - 1$, for $i = 0, \ldots, n-1$. Hence

$$L(S_0) = \sum_{i=0}^{n-1} a'_i \leq \sum_{i=0}^{n-1} (b_i + a_i - 1) = \sum_{i=0}^{n-1} b_i + \delta n. \tag{9}$$

By Lemma 4 in [16], the local optimality condition (5) implies $\sum_{i=0}^{n-1} b_i \leq n\sqrt{2}$. It follows that $L(S_0) \leq (\sqrt{2} + \delta)n$. Hence the star Steiner ratio is at most

$$\frac{L(S_0)}{L(SS_{\min})} \leq \frac{\sqrt{2} + \delta}{1 + \delta}. \tag{10}$$

Clearly, the sum of the lengths of the n stars (centered at each of the n points) equals twice the sum of pairwise distances among the points.

$$\sum_{i=0}^{n-1} L(S_i) = 2 \sum_{i<j} |p_i p_j|.$$

By the triangle inequality (see also Fig. 1(right))

$$|p_i p_j| \leq |p_i q_i| + |q_i q_j| + |q_j p_j| = (a_i - 1) + |q_i q_j| + (a_j - 1).$$

By summing over all pairs $i < j$, and using the upper bound on $G(d, n)$ in (1), we obtain

$$\sum_{i=0}^{n-1} L(S_i) \leq 2 \sum_{i<j} |q_i q_j| + 2(n-1) \sum_{i=0}^{n-1} (a_i - 1) = 2 \sum_{i<j} |q_i q_j| + 2\delta(n-1)n \leq (c_d + 2\delta)n^2.$$

The minimum of the n stars, $S_{\min}$, clearly satisfies:

$$L(S_{\min}) \leq \frac{1}{n} \sum_{i=0}^{n-1} L(S_i), \text{ hence } L(S_{\min}) \leq (c_d + 2\delta)n. \tag{11}$$

It follows that the star Steiner ratio is at most

$$\frac{L(S_{\min})}{L(SS_{\min})} \leq \frac{c_d + 2\delta}{1 + \delta}. \tag{12}$$

Inequalities (10) and (12) imply

$$\frac{L(S_{\min})}{L(SS_{\min})} \leq \max_{\delta \geq 0} \min \left(\frac{\sqrt{2} + \delta}{1 + \delta}, \frac{c_d + 2\delta}{1 + \delta} \right).$$

Observe that $\frac{\sqrt{2}+\delta}{1+\delta}$ is a monotonically decreasing function of δ, and $\frac{c_d+2\delta}{1+\delta}$ is a monotonically increasing function of δ, for every $1 < c_d < 2$. Their minimum is maximized for a value of δ satisfying $\frac{\sqrt{2}+\delta}{1+\delta} = \frac{c_d+2\delta}{1+\delta}$. The solution $\delta_0 = \sqrt{2} - c_d$ balances the two upper estimates in (10) and (12), and it yields

$$\rho_d \leq \rho_d(n) \leq \frac{2\sqrt{2} - c_d}{1 + \sqrt{2} - c_d}. \tag{13}$$

This completes the proof of Theorem 1. □

By substituting the values of c_d for $d = 2$ and $d = 3$, one obtains:

$$\frac{4}{\pi} \leq \rho_2 \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} < 1.3631,$$

and

$$\frac{4}{3} \leq \rho_3 \leq \frac{2}{17}(16 - 3\sqrt{2}) < 1.3833.$$

2.1 Minimum star to maximum matching ratios in two and three dimensions

First consider $d = 2$. Fekete and Meijer [16] showed that $L(SS_{\min}) \leq \frac{2}{\sqrt{3}} L(M_{\max})$. By our Theorem 1,

$$L(S_{\min}) \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} \cdot L(SS_{\min}).$$

Combining the two upper bounds yields the following new upper bound on η_2.

Corollary 1 *The maximum ratio η_2 between the minimum star and the maximum matching in the plane is less than 1.5739. That is, for any finite planar point set*

$$\frac{L(S_{\min})}{L(M_{\max})} \leq \frac{2\sqrt{2} - \frac{4}{\pi}}{1 + \sqrt{2} - \frac{4}{\pi}} \cdot \frac{2}{\sqrt{3}} \leq 1.5739.$$

The best known lower bound for this ratio, $4/3$, is given in [16]. It is obtained by placing $n/3$ points on each vertex of an equilateral triangle (for n divisible by 3). Note that this ratio can be approached arbitrarily close using distinct points.

Let now $d = 3$. Fekete and Meijer [16] showed that $L(SS_{\min}) \leq \sqrt{2} \cdot L(M_{\max})$. By our Theorem 2, $L(S_{\min}) \leq \frac{2}{17}(16 - 3\sqrt{2}) \cdot L(SS_{\min})$. Combining the two yields the following new upper bound on η_3.

Corollary 2 *The maximum ratio η_3 between the minimum star and the maximum matching in $\mathbb{R}^3$ is less than 1.9562. That is, for any finite point set in $\mathbb{R}^3$*

$$\frac{L(S_{\min})}{L(M_{\max})} \leq \frac{2}{17}(16 - 3\sqrt{2})\sqrt{2} = \frac{4}{17}(8\sqrt{2} - 3) < 1.9562.$$

The best known lower bound for this ratio, $3/2$, is given in [16]. It is obtained by placing $n/4$ points on each vertex of a regular tetrahedron (for n divisible by 4). Again, this ratio can be approached arbitrarily close using distinct points.

3 Asymptotic results and approximations

Theorem 2 *Let c_d be the constant of uniform density for the sphere in $\mathbb{R}^d$, for $d \geq 2$. Then*

$$\lim_{d\to\infty} c_d = \lim_{d\to\infty} \rho_d = \sqrt{2}.$$

The following closed formula approximations hold for $d \geq 3$:

$$\sqrt{2}e^{-\frac{1}{4(2d-3)}} \leq \rho_d \leq \frac{2\sqrt{2} - \sqrt{2}e^{-\frac{1}{5(2d-1)}}}{1 + \sqrt{2} - \sqrt{2}e^{-\frac{1}{5(2d-1)}}}.$$

Proof. In order to establish the limits, we start by computing the constant of uniform density c_d. Recall that c_d equals the average distance from a point on the unit sphere in $\mathbb{R}^d$ to all the other points on the same sphere. It is easy to verify that c_d is given by the following integral formula:

$$c_d = \frac{2\int_0^{\pi/2} \sin^{d-2}(2\alpha) \cdot \sin\alpha \, \mathrm{d}\alpha}{\int_0^{\pi/2} \sin^{d-2}(2\alpha) \, \mathrm{d}\alpha}, \quad \text{for} \geq 2. \tag{14}$$

Some initial values are

$$c_1 = 1, \;\; c_2 = \frac{4}{\pi} = 1.2732\ldots, \; c_3 = \frac{4}{3} = 1.3333\ldots,$$

$$c_4 = \frac{64}{15\pi} = 1.3581\ldots, \; c_5 = \frac{48}{35} = 1.3714\ldots.$$

In order to establish a recurrence on c_d, define

$$a_{ij} = \int_0^{\pi/2} \sin^i \alpha \cdot \cos^j \alpha \, \mathrm{d}\alpha, \;\; i, j \geq 0.$$

Some initial values are

$$a_{00} = \frac{\pi}{2}, \;\; a_{01} = a_{10} = 1, \;\; a_{11} = \frac{1}{2}, \;\; a_{02} = a_{20} = \frac{\pi}{4}.$$

Expanding $\sin 2\alpha$ yields then

$$c_d = \frac{2a_{d-1,d-2}}{a_{d-2,d-2}}.$$

Recall that integration by parts leads to the well-known recurrence relations for a_{ij}, for $i, j \geq 1$:

$$\begin{aligned} a_{ij} = \int_0^{\pi/2} \sin^i \alpha \cdot \cos^j \alpha \, \mathrm{d}\alpha &= -\frac{\sin^{i-1} \alpha \cdot \cos^{j+1} \alpha}{i+j}\Big|_0^{\pi/2} + \frac{i-1}{i+j} \int_0^{\pi/2} \sin^{i-2} \alpha \cdot \cos^j \alpha \, \mathrm{d}\alpha \\ &= \frac{\sin^{i+1} \alpha \cdot \cos^{j-1} \alpha}{i+j}\Big|_0^{\pi/2} + \frac{j-1}{i+j} \int_0^{\pi/2} \sin^{i} \alpha \cdot \cos^{j-2} \alpha \, \mathrm{d}\alpha. \end{aligned}$$

Plugging these in the formula for c_d immediately gives a recurrence for c_d. For any $d \geq 1$:

$$c_{d+2} = \frac{2 \cdot \frac{d}{2d+1} \cdot \frac{d-1}{2d-1} \cdot a_{d-1,d-2}}{\frac{d-1}{2d} \cdot \frac{d-1}{2d-2} \cdot a_{d-2,d-2}} = \frac{4d^2}{4d^2-1} c_d = \left(1 + \frac{1}{4d^2-1}\right) c_d.$$

Recall at this point the infinite Wallis product [17]:

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \left(\frac{4k^2}{4k^2-1}\right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots.$$

Let

$$W_n = \prod_{k=1}^{n} \left(\frac{4k^2}{4k^2-1}\right), \quad \text{and} \quad Z_n = \prod_{k=n+1}^{\infty} \left(\frac{4k^2}{4k^2-1}\right),$$

denote the partial finite and respectively partial infinite Wallis products, so that $W_n Z_n = \pi/2$, for every $n \geq 1$. Our recurrence for c_d yields that c_d is an increasing sequence satisfying also

$$c_{d+1} c_{d+2} = c_1 c_2 W_d, \quad \text{for } d \geq 1. \tag{15}$$

Since c_d is bounded, it converges to some limit c. The value of c can be obtained by solving the equation

$$c^2 = c_1 c_2 \frac{\pi}{2} = 2.$$

We thus have $\lim_{d\to\infty} c_d = \sqrt{2}$. Since $c_d \leq \rho_d \leq \sqrt{2}$, we also have $\lim_{d\to\infty} \rho_d = \sqrt{2}$. From Equation (15), we also get that for $d \geq 3$,

$$c_1 c_2 W_{d-2} \leq c_d^2 \leq c_1 c_2 W_{d-1} \quad \text{or} \quad \sqrt{c_1 c_2 W_{d-2}} \leq c_d \leq \sqrt{c_1 c_2 W_{d-1}}. \tag{16}$$

Observe that

$$\sum_{k=n+1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2} \sum_{k=n+1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = \frac{1}{2(2n+1)}.$$

Standard inequalities[1] $e^{4x/5} \leq 1 + x \leq e^x$ for $x \in [0, 1/3]$ now imply that for each $n \geq 1$

$$Z_n = \prod_{k=n+1}^{\infty} \left(1 + \frac{1}{4k^2-1}\right) \leq e^{\sum_{k=n+1}^{\infty} \frac{1}{4k^2-1}} = e^{\frac{1}{2(2n+1)}},$$

and

$$Z_n = \prod_{k=n+1}^{\infty} \left(1 + \frac{1}{4k^2-1}\right) \geq e^{\frac{4}{5}\sum_{k=n+1}^{\infty} \frac{1}{4k^2-1}} = e^{\frac{2}{5(2n+1)}}.$$

Since $W_n = (\pi/2)/Z_n$, we have

$$\frac{\pi}{2} \cdot e^{-\frac{1}{2(2n+1)}} \leq W_n \leq \frac{\pi}{2} \cdot e^{-\frac{2}{5(2n+1)}},$$

and consequently (16) gives

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} \cdot e^{-\frac{1}{4(2d-3)}} \leq c_d \leq \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} \cdot e^{-\frac{1}{5(2d-1)}},$$

or equivalently

$$\sqrt{2} e^{-\frac{1}{4(2d-3)}} \leq c_d \leq \sqrt{2} e^{-\frac{1}{5(2d-1)}}.$$

Taking into account (8) and substituting the above upper bound on c_d, we get the final estimate

$$\sqrt{2} e^{-\frac{1}{4(2d-3)}} \leq \rho_d \leq \frac{2\sqrt{2} - \sqrt{2} e^{-\frac{1}{5(2d-1)}}}{1 + \sqrt{2} - \sqrt{2} e^{-\frac{1}{5(2d-1)}}}, \quad \text{for } d \geq 3.$$

The proof of Theorem 2 is now complete. □

4 Verification of Conjecture 1 in two special cases

We show that the bound $c_2 = \frac{4}{\pi}$ in $\mathbb{R}^2$ is best possible in two special cases, corresponding to the lower bound construction. Specifically, we show that the star Steiner ratio is at most $\frac{\pi/2n}{\tan(\pi/2n)} \cdot \frac{4}{\pi} < \frac{4}{\pi}$ (i) for any finite point set on a circle centered at the Weber point in $\mathbb{R}^2$, and (ii) for any finite point set in the plane where the angles from the Weber center to the n points are uniformly distributed (that is, $\alpha_i = 2i\pi/n$, for $i = 0, 1, \ldots, n-1$). The two tight bounds in the plane essentially follow from the closed formula (2) for $G(2, n)$ due to László Fejes Tóth [15] and mentioned in the Introduction. We also give suitable generalizations of these two planar results for points in $\mathbb{R}^d$.

[1]Here we have chosen $4x/5$ to simplify the resulting expressions.

Theorem 3 *Let $P = \{p_0, p_1, \ldots, p_{n-1}\}$ be set of n (not necessarily distinct) points in $\mathbb{R}^d$ such that their Weber point is the origin o, and a closest point to o in P is p_0, at unit distance from o. For $i = 0, 1, \ldots, n-1$, let q_i be the intersection point of line segment op_i and the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$; in particular, $q_0 = p_0$. If the star over $Q = \{q_0, q_1, \ldots, q_{n-1}\}$ centered at q_0 has length $\sum_{i=1}^{n-1} |q_o q_i| \leq \frac{2}{n} \cdot G(d, n)$, then the star Steiner ratio of P is at most $\frac{2}{n^2} \cdot G(d, n) \leq c_d$.*

Proof. Let $a_i = |op_i|$, for $i = 0, 1, \ldots, n-1$, as in our general setup in Section 2. We have $L(SS_{\min}) = \sum_{i=0}^{n-1} a_i = (1+\delta)n = n + \delta n$, for some $\delta \geq 0$.

By the triangle inequality, $|p_0 p_i| \leq |p_0 q_i| + |q_i p_i| = |q_0 q_i| + (a_i - 1)$. Therefore, the length of the star over P centered at p_0 is

$$\sum_{i=0}^{n-1} |p_0 p_i| \leq \sum_{i=0}^{n-1} |q_0 q_i| + \sum_{i=0}^{n-1} (a_i - 1) \leq \frac{2G(d,n)}{n} + \delta n.$$

It is easy to see that $G(d, n) \geq \frac{n^2}{2}$, or equivalently, $\frac{2}{n} \cdot G(d, n) \geq n$. Using this and the previous inequality, we deduce that the star Steiner ratio in this special case is:

$$\frac{2G(d,n)/n + \delta n}{n + \delta n} \leq \frac{2G(d,n)}{n^2} \leq c_d. \qquad \square$$

Corollary 3 (i) *The star Steiner ratio for a set of n points in $\mathbb{R}^d$ that lie on a sphere centered at the Weber center is at most $\frac{2}{n^2} \cdot G(d, n) \leq c_d$.*
(ii) *In particular, the star Steiner ratio for a set of n points in the plane that lie on a circle centered at the Weber center is at most*

$$\frac{2G(2,n)}{n^2} = \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \cdot \frac{4}{\pi} < \frac{4}{\pi} = c_2.$$

Proof. (i) We may assume that the points in P lie on a *unit* sphere $\mathbb{S}^{d-1}$ centered at the origin. The total length of the n stars for P is at most $2G(d, n)$, by the definition of $G(d, n)$. By using the first inequality in (11), there is a point $p \in P$ such that the length of the star centered at p is at most $\frac{2}{n} \cdot G(d, n)$. An application of Theorem 3 with $p_0 = p$ completes the proof.
(ii) By the closed formula (2), we have $G(2, n) = \frac{n}{\tan \frac{\pi}{2n}}$, and the bound follows as in part (i). $\square$

Corollary 4 *The star Steiner ratio for a set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ of n points in the plane where point p_i is visible from the origin under angle $2i\pi/n$, for $i = 0, 1, \ldots, n-1$, is at most*

$$\frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \cdot \frac{4}{\pi} < \frac{4}{\pi}.$$

Proof. [2] We may assume w.l.o.g. that $p_0 = (1, 0)$ is a closest point in P to the origin o. By equation (5), the Weber center of P is the origin o. For $i = 0, 1, \ldots, n-1$, let q_i be the intersection point of line segment op_i and the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$; in particular, $q_0 = p_0$.

Note that the point set Q is symmetric under rotation by $2\pi/n$ about the origin. It follows that every star of Q has the same length. The total length of all stars of Q is at most $2G(2, n)$, by the definition of $G(2, d)$. Hence, the length of the star of Q centered at q_0 is at most $\frac{2}{n} \cdot G(2, n)$. An application of Theorem 3 completes the proof. $\square$

[2] The earlier proof of this result in [12] had an error.

5 Conclusion

We think that the sharper upper bound on the star Steiner ratio in the plane in the two special cases (in Corollary 3 and Corollary 4) always holds, so we venture here a slightly stronger version of the conjecture proposed by Fekete and Meijer [16]:

Conjecture 1 *The star Steiner ratio for n points in the plane is*

$$\rho_2(n) = \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} \cdot \frac{4}{\pi}.$$

Fekete and Meijer [16] conjectured that $\rho_2 = 4/\pi$ and $\rho_3 = 4/3$, where these values coincide with c_2 and c_3, respectively. We extend their conjecture to all dimensions $d \geq 2$:

Conjecture 2 *The star Steiner ratio in $\mathbb{R}^d$ equals the constant of uniform density for the sphere in $\mathbb{R}^d$, that is, $\rho_d = c_d$ for every $d \geq 2$.*

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