Minimum-perimeter intersecting polygons*

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Abstract

Given a set $\mathcal{S}$ of segments in the plane, a polygon P is an intersecting polygon of $\mathcal{S}$ if every segment in $\mathcal{S}$ intersects the interior or the boundary of P. The problem MPIP of computing a minimum-perimeter intersecting polygon of a given set of n segments in the plane was first considered by Rappaport in 1995. This problem is not known to be polynomial, nor it is known to be NP-hard. Rappaport (1995) gave an exponential-time exact algorithm for MPIP. Hassanzadeh and Rappaport (2009) gave a polynomial-time approximation algorithm with ratio $\frac{\pi}{2} \approx 1.57$. In this paper, we present two improved approximation algorithms for MPIP: a 1.28-approximation algorithm by linear programming, and a polynomial-time approximation scheme by discretization and enumeration. Our algorithms can be generalized for computing an approximate minimum-perimeter intersecting polygon of a set of convex polygons in the plane. From the other direction, we show that computing a minimum-perimeter intersecting polygon of a set of (not necessarily convex) simple polygons is NP-hard.

Keywords: convex polygon, approximation algorithm, linear programming, NP-hardness.

1 Introduction

A polygon P is an *intersecting polygon* of a set $\mathcal{S}$ of segments in the plane if every segment in $\mathcal{S}$ intersects the interior or the boundary of P. In 1995, Rappaport [12] proposed the following geometric optimization problem; see Figure 1 for an example.

MPIP: Given a set $\mathcal{S}$ of n (possibly intersecting) segments in the plane, compute a minimum-perimeter intersecting polygon P^* of $\mathcal{S}$.

The problem MPIP was originally motivated by the theory of geometric transversals; see [14] for a recent survey on related topics. As of now, MPIP is not known to be solvable in polynomial time, nor it is known to be NP-hard. Rappaport [12] gave an exact algorithm for MPIP that runs in $O(n \log n)$ time when the input segments are constrained to a constant number of orientations, but the running time becomes exponential in the general case. Recently, Hassanzadeh and Rappaport [8] presented the first polynomial-time constant-factor approximation algorithm for MPIP with ratio $\frac{\pi}{2} = 1.57\ldots$.

In this paper, we present two improved approximation algorithms for MPIP. Our first result (in Section 2) is a 1.28-approximation algorithm based on linear programming:

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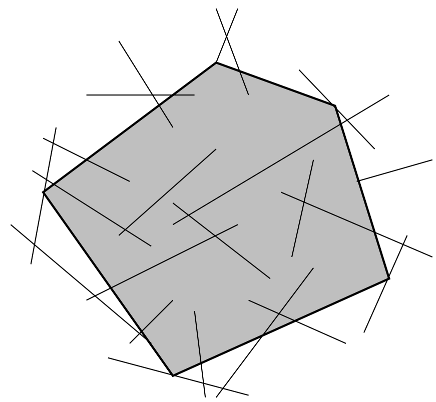

Figure 1: A minimum-perimeter intersecting polygon (drawn with thick edges) for a set of segments.

Theorem 1. *For any $\varepsilon > 0$, a $\frac{4}{\pi}(1+\varepsilon)$-approximation for minimum-perimeter intersecting polygon of n segments in the plane can be computed by solving $O(1/\varepsilon)$ linear programs, each with $O(n)$ variables and $O(n)$ constraints. In particular, a 1.28-approximation can be computed by solving a constant number of such linear programs.*

Our second result (in Section 3) is a polynomial-time approximation scheme (PTAS) based on discretization and enumeration:

Theorem 2. *For any $\varepsilon > 0$, a $(1 + \varepsilon)$-approximation for minimum-perimeter intersecting polygon of n segments in the plane can be computed in $O(1/\varepsilon)\,\mathrm{poly}(n) + 2^{O((1/\varepsilon)^{2/3})}n$ time.*

While the problem MPIP has been initially formulated for segments, it can be formulated more generally as the problem of computing a minimum-perimeter intersecting polygon for any finite collection of connected (say, polygonal) regions in the plane. Natural variants to consider are for a set of convex or non-convex polygons, and for a set of polygonal chains. In Section 4, we show that both algorithms in Theorem 1 and Theorem 2 can be generalized for computing an approximate minimum-perimeter intersecting polygon of a set of (possibly intersecting) convex polygons in the plane. From the other direction, we show in Section 5 that computing a minimum-perimeter intersecting polygon of a set of (possibly intersecting but not necessarily convex) simple polygons is NP-hard:

Theorem 3. *Computing a minimum-perimeter intersecting polygon of a set of simple polygons, or that of a set of simple polygonal chains, is NP-hard.*

Preliminaries. Let P^* denote a minimum-perimeter intersecting polygon of $\mathcal{S}$. Denote by $\mathrm{perim}(P)$ the perimeter of a polygon P. We can assume without loss of generality that not all segments in $\mathcal{S}$ are concurrent at a common point (this can be easily checked in linear time), thus $\mathrm{perim}(P^*) > 0$. The following two facts are easy to prove; see also [8, 12].

Proposition 1. *P^* is a convex polygon with at most n vertices.*

Proposition 2. *If P is an intersecting polygon of $\mathcal{S}$, and P is contained in another polygon P', then P' is also an intersecting polygon of $\mathcal{S}$.*

2 A $\frac{4}{\pi}(1+\varepsilon)$-approximation algorithm

In this section we prove Theorem 1. We present a $\frac{4}{\pi}(1+\varepsilon)$-approximation algorithm for computing a minimum-perimeter intersecting polygon of a set $\mathcal{S}$ of line segments. The idea is to first show that every convex polygon P is contained in some rectangle $R = R(P)$ that satisfies $\mathrm{perim}(R) \leq \frac{4}{\pi}\,\mathrm{perim}(P)$, then use linear programming to compute a $(1+\varepsilon)$-approximation for the minimum-perimeter intersecting rectangle of $\mathcal{S}$. To implement the second step above, we enumerate many discrete directions α_i, that cover the entire range of directions, and use linear programming to compute the minimum-perimeter intersecting rectangle R_i of $\mathcal{S}$ in each direction α_i.

Algorithm A1.

> Let $m = \lceil \frac{\pi}{4\varepsilon} \rceil$. For each direction $\alpha_i = i \cdot 2\varepsilon$, $i = 0, 1, \ldots, m-1$, compute a minimum-perimeter intersecting rectangle R_i of $\mathcal{S}$ with orientation α_i, $0 \leq \alpha_i < \pi/2$. Return the rectangle with the minimum perimeter over all m directions.

We now show how to compute the rectangle R_i by linear programming. By a suitable rotation of the set $\mathcal{S}$ of segments in each iteration $i \geq 1$, we can assume for convenience that the rectangle R_i is axis-parallel, that is, $R_i = [x_1, x_2] \times [y_1, y_2]$ for four variables x_1, x_2, y_1, y_2. Then the objective of minimum perimeter is easily expressed as a linear function of the four variables. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be the input set of n segments, and let $u_j = (a_j, b_j)$ and $v_j = (c_j, d_j)$ be the two endpoints of S_j. Note that a point p_j belongs to the segment S_j if and only if it is a convex combination of the two endpoints u_j and v_j, that is, $p_j = (1-t_j)u_j + t_j v_j$ for some variable $t_j \in [0,1]$. To satisfy the intersection requirement, each segment S_j in $\mathcal{S}$ must have a point p_j in the rectangle R_i. This can be expressed by linear constraints of the variable t_j and the four variables x_1, x_2, y_1, y_2. Thus we have the following linear program with $n+4$ variables and $6n$ constraints:

$$
\begin{aligned}
&\text{minimize} \quad 2(x_2 - x_1) + 2(y_2 - y_1) && \text{(LP1)}\\
&\text{subject to} \quad \begin{cases} x_1 \leq (1-t_j)a_j + t_j c_j \leq x_2, & 1 \leq j \leq n \\ y_1 \leq (1-t_j)b_j + t_j d_j \leq y_2, & 1 \leq j \leq n \\ 0 \leq t_j \leq 1, & 1 \leq j \leq n \end{cases}
\end{aligned}
$$

A key fact in the analysis of the algorithm is the following lemma. This inequality is also implicit in [13], where a slightly different proof is given. Nevertheless we present here our own proof for completeness. Recall that a *convex body* is a compact convex set with nonempty interior.

Lemma 1. *Let R be the minimum-perimeter rectangle that contains a planar convex body B. Then $\mathrm{perim}(R) \leq \frac{4}{\pi}\,\mathrm{perim}(B)$. This inequality is tight.*

Proof. Refer to Figure 2. For $\alpha \in [0, \pi/2)$, let $R(\alpha)$ be the minimum-perimeter rectangle with orientation α that contains B. Since $\mathrm{perim}(R) \leq \mathrm{perim}(R(\alpha))$ for every $\alpha \in [0, \pi/2)$, we have

$$\frac{\pi}{2} \cdot \mathrm{perim}(R) \leq \int_0^{\pi/2} \mathrm{perim}(R(\alpha))\,\mathrm{d}\alpha. \tag{1}$$

For $\alpha \in [0, \pi)$, let $w(\alpha)$ denote the minimum width of a parallel strip along the direction α that contains B. Note that $\mathrm{perim}(R(\alpha)) = 2w(\alpha) + 2w(\alpha + \pi/2)$. It follows that

$$\int_0^{\pi/2} \mathrm{perim}(R(\alpha))\,\mathrm{d}\alpha = 2\int_0^{\pi/2} w(\alpha)\,\mathrm{d}\alpha + 2\int_0^{\pi/2} w(\alpha + \pi/2)\,\mathrm{d}\alpha = 2\int_0^{\pi} w(\alpha)\,\mathrm{d}\alpha. \tag{2}$$

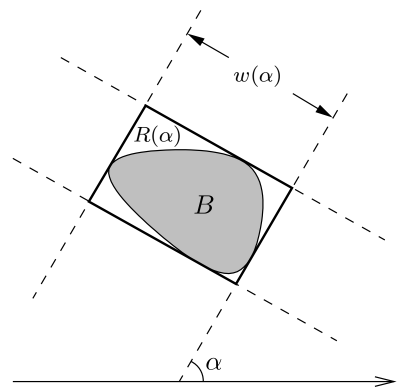

Figure 2: The rectangle $R(\alpha)$ is the intersection of two strips of widths $w(\alpha)$ and $w(\alpha + \pi/2)$ along the directions α and $\alpha + \pi/2$, respectively.

According to Cauchy's surface area formula [11, pp. 283–284],

$$\int_0^\pi w(\alpha)\,\mathrm{d}\alpha = \mathrm{perim}(B). \tag{3}$$

From (1), (2) and (3), we have

$$\frac{\pi}{2} \cdot \mathrm{perim}(R) \le \int_0^{\pi/2} \mathrm{perim}(R(\alpha))\,\mathrm{d}\alpha = 2\int_0^\pi w(\alpha)\,\mathrm{d}\alpha = 2 \cdot \mathrm{perim}(B),$$

which gives the claimed inequality $\mathrm{perim}(R) \le \frac{4}{\pi} \cdot \mathrm{perim}(B)$. Observe that the inequality becomes equality if B is a disk. □

Let R^* be a minimum-perimeter intersecting rectangle of $\mathcal{S}$. Let P^* be a minimum-perimeter intersecting polygon of $\mathcal{S}$. It follows from Lemma 1 that

$$\mathrm{perim}(R^*) \le \frac{4}{\pi}\,\mathrm{perim}(P^*). \tag{4}$$

To account for the error caused by discretization, we need the following lemma:

Lemma 2. *For all* $i = 0, 1, \ldots, m-1$, $\mathrm{perim}(R_i) \le \sqrt{2}\,\mathrm{perim}(R^*)$. *Moreover, there exists an* $i \in \{0, 1, \ldots, m-1\}$ *such that* $\mathrm{perim}(R_i) \le (1+\varepsilon)\,\mathrm{perim}(R^*)$.

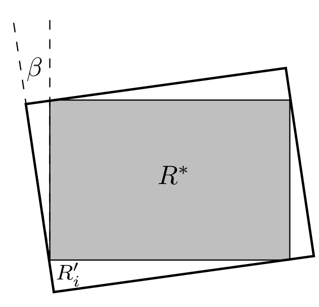

Figure 3: Discretization error due to the angle difference β between the orientations of R_i and R^*.

Proof. Refer to Figure 3. Consider any rectangle R_i, $i \in \{0, 1, \ldots, m-1\}$. Let β be the minimum angle difference between the orientations of R_i and R^*, $0 \le \beta \le \pi/4$. Let R_i' be the minimum-perimeter rectangle with the same orientation as R_i such that R_i' contains R^*. An easy trigonometric calculation shows that

$$\operatorname{perim}(R_i') = (\cos\beta + \sin\beta)\operatorname{perim}(R^*).$$

It follows that

$$\operatorname{perim}(R_i) \le \operatorname{perim}(R_i') = (\cos\beta + \sin\beta)\operatorname{perim}(R^*) \le \sqrt{2}\operatorname{perim}(R^*).$$

Since the directions $\alpha_i = i \cdot 2\varepsilon$ are discretized with consecutive difference at most 2ε, there exists an $i \in \{0, 1, \ldots, m\}$ such that the angle β between the orientations of R_i and R^* is at most ε. For this i, we have

$$\operatorname{perim}(R_i) \le (\cos\beta + \sin\beta)\operatorname{perim}(R^*) \le (1+\beta)\operatorname{perim}(R^*) \le (1+\varepsilon)\operatorname{perim}(R^*),$$

as required. □

Let R_i be the rectangle returned by Algorithm A1. It follows from Lemma 2 that

$$\operatorname{perim}(R_i) \le (1+\varepsilon)\operatorname{perim}(R^*). \tag{5}$$

Combining the two inequalities (4) and (5), we have

$$\operatorname{perim}(R_i) \le (1+\varepsilon)\operatorname{perim}(R^*) \le \frac{4}{\pi}(1+\varepsilon)\operatorname{perim}(P^*).$$

This completes the proof of Theorem 1.

3 A polynomial-time approximation scheme

In this section we prove Theorem 2. We present a $(1+\varepsilon)$-approximation algorithm for computing a minimum-perimeter intersecting polygon of a set $\mathcal{S}$ of line segments. The idea is to first locate a region Q that contains either an optimal polygon P^* or a good approximation of it, then enumerate a suitable set of convex grid polygons contained in Q to approximate P^*.

Algorithm A2.

Step 1. Let $\varepsilon_1 = \frac{\varepsilon}{2+\varepsilon}$. Run Algorithm A1 to compute a rectangle R that is a $(1+\varepsilon_1)$-approximation of the minimum-perimeter intersecting rectangle of $\mathcal{S}$. Let Q be a square of side length $3\operatorname{perim}(R)$ that is concentric with R and parallel to R.

Step 2. Let $k = \lceil 48/\varepsilon \rceil$. Divide the square Q into a $k \times k$ grid Q_δ of cell length $\delta = 3\operatorname{perim}(R)/k$. Enumerate all convex grid polygons with grid vertices from Q_δ. Find an intersecting polygon P_δ with the minimum perimeter among these grid polygons.

Step 3. Return R or P_δ, the one with the smaller perimeter.

Define the *distance* between two compact sets A and B in the plane as the minimum distance between two points $a \in A$ and $b \in B$; A and B intersect if and only if their distance is zero. For a planar convex body A, denote by $x_{\min}(A)$ and $x_{\max}(A)$, respectively, the minimum and the maximum x-coordinates of a point in A.

Let P^* be a minimum-perimeter intersecting polygon of $\mathcal{S}$ at the *smallest* distance to R. The choice of the square region Q in Step 1 of Algorithm A2 is justified by the following lemma:

Lemma 3. *Suppose that* $\mathrm{perim}(R) \geq (1+\varepsilon)\,\mathrm{perim}(P^*)$*. Then* $P^* \subseteq Q$*.*

Proof. Suppose first that R and P^* are disjoint. Refer to Figure 4. Consider the two common supporting lines of R and P^*, each tangent to both R and P^* on one side. Assume without loss of generality that the two lines are symmetric about the x axis. Assume without loss of generality that $x_{\max}(R) \leq x_{\max}(P^*)$. We next consider two cases: (1) $x_{\max}(R) < x_{\min}(P^*)$, and (2) $x_{\max}(R) \geq x_{\min}(P^*)$.

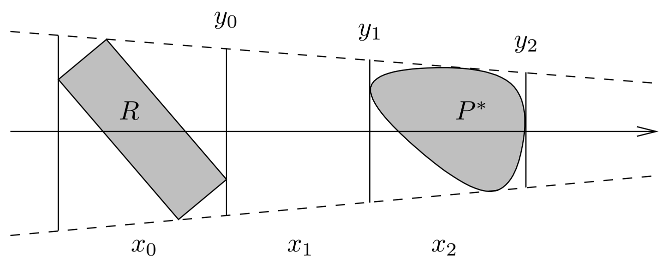

Figure 4: The two dashed lines symmetric about the x axis are tangent to both R and P^*. The four vertical segments connecting the two lines are inter-spaced by x_0, x_1, and x_2. The first and second segments are tangent to R, while the third and fourth segments are tangent to P^*. The second, third, and fourth segments have lengths y_0, y_1, and y_2, respectively.

Case 1: $x_{\max}(R) < x_{\min}(P^*)$. Refer to Figure 4. The crucial observation in this case is:

($\star$) Since both R and P^* are intersecting polygons of $\mathcal{S}$, each segment in $\mathcal{S}$ has at least one point in R and one point in P^*.

By the convexity of each segment in $\mathcal{S}$, any vertical segment between the two supporting lines with x-coordinate at least $x_{\max}(R)$ and at most $x_{\min}(P^*)$ is a degenerate intersecting rectangle of $\mathcal{S}$. For example, in Figure 4, the two vertical segments of lengths y_0 and y_1 correspond to two degenerate intersecting rectangles of perimeters $2y_0$ and $2y_1$, respectively. In particular, $\mathrm{perim}(P^*) \leq 2y_1$. By our choice of P^*, we must have $y_0 > y_1$, because otherwise there would be an intersecting polygon of perimeter $2y_0 \leq 2y_1 \leq \mathrm{perim}(P^*)$ that is closer to R than P^* is.

Recall that R is a $(1+\varepsilon_1)$-approximation for the minimum-perimeter intersecting rectangle of $\mathcal{S}$. Thus we have $\mathrm{perim}(R) \leq (1+\varepsilon_1)\mathrm{perim}(P^*) \leq (1+\varepsilon_1)\,2y_1$. Since $\mathrm{perim}(R) \geq 2y_0$, it follows that $y_0 \leq (1+\varepsilon_1)y_1$. Also, by the assumption of the lemma, we have

$$\mathrm{perim}(P^*) \leq \frac{1}{1+\varepsilon}\,\mathrm{perim}(R) = \frac{1-\varepsilon_1}{1+\varepsilon_1}\,\mathrm{perim}(R) \leq (1-\varepsilon_1)\,2y_1.$$

Since $\mathrm{perim}(P^*) \geq 2y_2$, it follows that $y_2 \leq (1-\varepsilon_1)y_1$.

Let $x_0 = x_{\max}(R) - x_{\min}(R)$, $x_1 = x_{\min}(P^*) - x_{\max}(R)$, and $x_2 = x_{\max}(P^*) - x_{\min}(P^*)$. It is clear that $x_0 \leq \frac{1}{2}\,\mathrm{perim}(R)$. By triangle similarity, we have

$$\frac{x_1}{x_2} = \frac{y_0 - y_1}{y_1 - y_2} \leq \frac{(1+\varepsilon_1)y_1 - y_1}{y_1 - (1-\varepsilon_1)y_1} = 1.$$

Thus $x_1 \leq x_2 \leq \frac{1}{2}\,\mathrm{perim}(P^*) \leq \frac{1}{2}\,\mathrm{perim}(R)$.

Let R' be the smallest axis-parallel rectangle that contains P^* and is concentric with R. Then the width of R' is at most $x_0 + 2(x_1 + x_2) \leq \frac{5}{2}\,\mathrm{perim}(R)$, and the height of R' is at most $y_1 < y_0 \leq \frac{1}{2}\,\mathrm{perim}(R)$. Since $(\frac{5}{2})^2 + (\frac{1}{2})^2 < 3^2$, the axis-parallel rectangle R' is contained in the square Q of

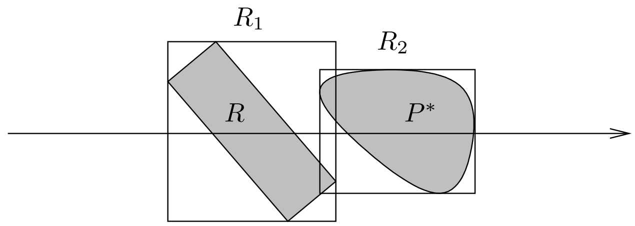

Figure 5: The two rectangles R_1 and R_2 intersect when $x_{\max}(R) \geq x_{\min}(P^*)$.

side length $3\,\text{perim}(R)$ that is concentric with R and parallel to R (recall that R is not necessarily axis-parallel). Thus $P^* \subseteq Q$, as required.

Case 2: $x_{\max}(R) \geq x_{\min}(P^*)$. Refer to Figure 5. Let R_1 be the smallest axis-parallel rectangle that contains R. Let R_2 be the smallest axis-parallel rectangle that contains P^*. Then the two rectangles R_1 and R_2 intersect. Let ℓ_1 and ℓ_2 be the maximum side lengths of R_1 and R_2, respectively. Then R_2 is contained in an axis-parallel square Q' of side length $\ell_1 + 2\ell_2$ that is concentric with R_1 (and R). Note that $\ell_1 \leq \frac{1}{2}\,\text{perim}(R)$ and $\ell_2 \leq \frac{1}{2}\,\text{perim}(P^*) \leq \frac{1}{2}\,\text{perim}(R)$. Thus $\ell_1 + 2\ell_2 \leq \frac{3}{2}\,\text{perim}(R)$. Since $(\frac{3}{2})^2 + (\frac{3}{2})^2 < 3^2$, this axis-parallel square Q' is contained in the square Q of side length $3\,\text{perim}(R)$ that is concentric with R and parallel to R (recall that R is not necessarily axis-parallel). Thus again $P^* \subseteq Q$.

Finally we consider that case when R intersects P^*. Define the two axis-parallel rectangles R_1 and R_2 as in Case 2. Then again the two rectangles R_1 and R_2 intersect, and the same argument proves that $P^* \subseteq Q$. This completes the proof of the lemma. □

The discretization error of convex grid polygons in Q is handled by the following lemma:

Lemma 4. *Suppose that $P^* \subseteq Q$. Then there exists a convex grid polygon P_δ with grid vertices from Q_δ such that $P^* \subseteq P_\delta$ and* $\text{perim}(P_\delta) \leq (1+\varepsilon)\,\text{perim}(P^*)$.

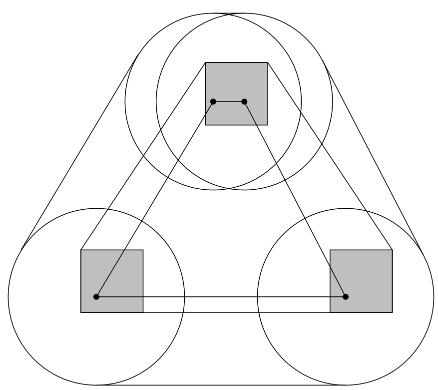

Figure 6: V (dots), C (shaded squares), and D (disks).

Proof. Denote by $\text{conv}(X)$ the convex hull of a planar set X. Refer to Figure 6. Let V be the set of vertices of the polygon P^*. Let C be the union of the grid cells of Q_δ that contain the vertices in V. Let D be the union of the disks of radii $r = \sqrt{2}\delta$ centered at the vertices in V. Then $V \subseteq C \subseteq D$. It follows that $\text{conv}(V) \subseteq \text{conv}(C) \subseteq \text{conv}(D)$. Note that $P^* = \text{conv}(V)$. Let $P_\delta = \text{conv}(C)$. Then

$P^* = \mathrm{conv}(V) \subseteq \mathrm{conv}(C) = P_\delta$. We also have

$$\mathrm{perim}(P_\delta) = \mathrm{perim}(\mathrm{conv}(C)) \le \mathrm{perim}(\mathrm{conv}(D)) = \mathrm{perim}(\mathrm{conv}(V)) + 2\pi r = \mathrm{perim}(P^*) + 2\pi\sqrt{2}\delta.$$

Recall that $\delta = 3\,\mathrm{perim}(R)/k = 3\,\mathrm{perim}(R)/\lceil 48/\varepsilon \rceil \le \frac{\varepsilon}{16}\,\mathrm{perim}(R)$. Let R^* be a minimum-perimeter intersecting rectangle of $\mathcal{S}$. By Lemma 2 and Lemma 1 in the previous section, we have $\mathrm{perim}(R) \le \sqrt{2}\,\mathrm{perim}(R^*)$ and $\mathrm{perim}(R^*) \le \frac{4}{\pi}\,\mathrm{perim}(P^*)$. Thus

$$2\pi\sqrt{2}\delta \le 2\pi\sqrt{2} \cdot \frac{\varepsilon}{16} \cdot \sqrt{2} \cdot \frac{4}{\pi} \cdot \mathrm{perim}(P^*) = \varepsilon \cdot \mathrm{perim}(P^*).$$

So we have $\mathrm{perim}(P_\delta) \le (1+\varepsilon)\,\mathrm{perim}(P^*)$. □

By Lemma 3 and Lemma 4, Algorithm A2 indeed computes a $(1+\varepsilon)$-approximation for the minimum-perimeter intersecting polygon of $\mathcal{S}$. We now analyze its time complexity. Step 1 runs in $O(1/\varepsilon_1)\,\mathrm{poly}(n) = O(1/\varepsilon)\,\mathrm{poly}(n)$ time. In Step 2, each convex grid polygon in a $k \times k$ grid has $O(k^{2/3})$ grid vertices [1, 2]. It follows from a result of Bárány and Pach [4] that there are $2^{O(k^{2/3})}$ such polygons in a $k \times k$ grid. Moreover, all these polygons can be enumerated in $2^{O(k^{2/3})}$ time since the proof in [4] is constructive. For each convex grid polygon, computing its perimeter takes $O(1/\varepsilon)$ time, and checking whether it is an intersecting polygon of $\mathcal{S}$ takes $O(n/\varepsilon)$ time, by simply checking each segment for intersection in $O(1/\varepsilon)$ time. Thus Step 2 runs in $2^{O((1/\varepsilon)^{2/3})}O(n/\varepsilon) = 2^{O((1/\varepsilon)^{2/3})}n$ time, and the total running time of Algorithm A2 is $O(1/\varepsilon)\,\mathrm{poly}(n) + 2^{O((1/\varepsilon)^{2/3})}n$.

4 Generalization for convex polygons

Both Algorithm A1 and Algorithm A2 can be generalized for computing an approximate minimum-perimeter intersecting polygon of a set $\mathcal{C}$ of n (possibly intersecting) convex polygons in the plane.

To generalize Algorithm A1, for each direction $\alpha_i = i \cdot 2\varepsilon$, $i = 0, 1, \ldots, m-1$, we simply replace the linear program LP1 by another linear program LP2, to compute a minimum-perimeter intersecting rectangle R_i of $\mathcal{C}$ with orientation α_i. As earlier, by a suitable rotation of the set $\mathcal{C}$ of polygons in each iteration $i \ge 1$, we can assume that the rectangle R_i is axis-parallel, that is, $R_i = [x_1, x_2] \times [y_1, y_2]$ for four variables x_1, x_2, y_1, y_2. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be the input set of n polygons, and let n_j be the number of vertices in C_j. Each convex polygon C_j can be represented as the intersection of n_j linear constraints (halfplanes). To satisfy the intersection requirement, each convex polygon C_j in $\mathcal{C}$ must have a point p_j in the rectangle R_i. Thus we have a linear program written symbolically as follows:

$$\begin{aligned} &\text{minimize} \quad 2(x_2 - x_1) + 2(y_2 - y_1) \\ &\text{subject to} \quad \begin{cases} p_j \in C_j, & 1 \le j \le n \\ p_j \in R_i, & 1 \le j \le n \end{cases} \end{aligned} \tag{LP2}$$

The linear program LP2 has $2n+4$ variables and $4n + \sum_j n_j$ constraints: There are $2n$ variables for the point coordinates, $p_j = (s_j, t_j)$, $1 \le j \le n$, and 4 variables x_1, x_2, y_1, y_2 for the rectangle $R_i = [x_1, x_2] \times [y_1, y_2]$. There are n_j linear constraints corresponding to $p_j \in C_j$, and 4 linear constraints corresponding to $p_j \in R_i$.

To generalize Algorithm A2, use the generalized Algorithm A1 in Step 1, then in Step 2 replace the checking for intersection between a convex polygon and a segment by the checking for intersection between two convex polygons. Lemma 3 is still valid because the crucial observation ($\star$) remains valid when segments are replaced by convex polygons; since convexity is a property shared by segments and convex polygons, the argument based on this observation continues to hold. Lemma 4 is utterly unaffected by the generalization.

5 NP-hardness

In this section we prove Theorem 3. We show that computing a minimum-perimeter intersecting polygon of a set of simple polygons, or that of a set of simple polygonal chains, is NP-hard. We make a reduction from the NP-hard problem *Vertex Cover* [7]:

INSTANCE: A graph $G = (V, E)$ with a set V of n vertices and a set E of m edges, and a positive integer $k < n$.

QUESTION: Is there a subset $S \subseteq V$ of k vertices such that S contains at least one vertex from each edge in E?

For simplicity, we describe our reduction for a set of simple polygonal chains. Note that a simple polygonal chain is a degenerate simple polygon with zero area; by slightly "fattening" the polygonal chains, our reduction also works for simple polygons.

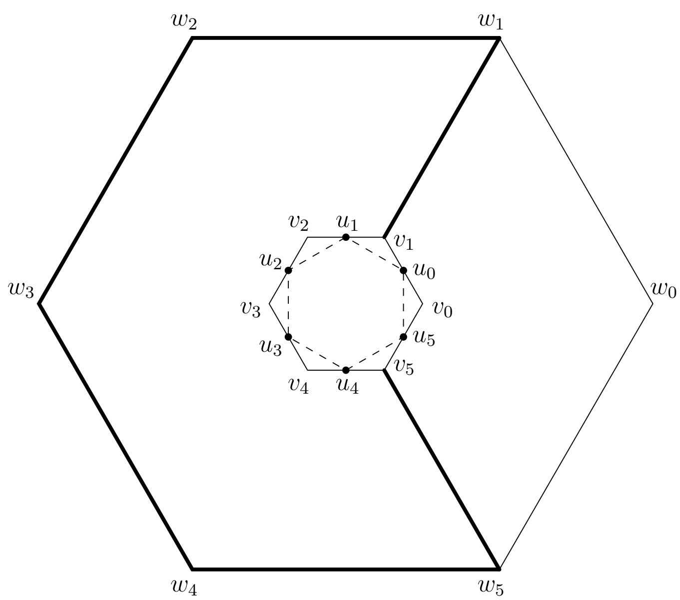

Figure 7: Reduction from vertex cover. For $n = 6$, $V_6 = v_0v_1v_2v_3v_4v_5$ and $W_6 = w_0w_1w_2w_3w_4w_5$ are two regular hexagons centered at the origin. The vertices of the dashed hexagon $U_6 = u_0u_1u_2u_3u_4u_5$ are midpoints of the edges of V_6. The edge $\{1, 5\}$ is represented by the polygonal chain $v_1w_1w_2w_3w_4w_5v_5$.

Assume that $n \geq 5$. Given a graph $G = (V, E)$, we construct a set $\mathcal{Z}$ of $n + m$ polygonal chains. Refer to Figure 7. Let $V = \{0, 1, \ldots, n-1\}$. Let V_n be a regular n-gon centered at the origin, with vertices $v_i = (\cos \frac{i \cdot 2\pi}{n}, \sin \frac{i \cdot 2\pi}{n})$, $i = 0, 1, \ldots, n-1$. Let W_n be another regular n-gon centered at the origin, with vertices $w_i = (4 \cos \frac{i \cdot 2\pi}{n}, 4 \sin \frac{i \cdot 2\pi}{n})$, $i = 0, 1, \ldots, n-1$. Let $U_n = u_0u_1 \ldots u_{n-1}$ be a regular n-gon inscribed in V_n such that the vertices of U_n are the midpoints of the edges of V_n, that is, $u_i = \frac{1}{2}(v_i + v_{i+1 \bmod n})$, $i = 0, 1, \ldots, n-1$. The set $\mathcal{Z}$ includes n polygonal chains that degenerate into the n points u_i, $i = 0, 1, \ldots, n-1$, and m polygonal chains that represent the m edges in E, where each edge $\{i, j\}$ in E, $0 \leq i < j \leq n-1$, is represented by the polygonal chain $v_iw_iw_{i+1} \ldots w_jv_j$ in $\mathcal{Z}$.

Define

$$f_n(k) = 2n \sin \frac{\pi}{n} \cos \frac{\pi}{n} + 2k \sin \frac{\pi}{n} \left(1 - \cos \frac{\pi}{n}\right). \tag{6}$$

Note that $f_n(k)$ is increasing in k. Next we prove two technical lemmas:

Lemma 5. *Let H be the convex hull of the n vertices of U_n and any k vertices of V_n. Then* $\mathrm{perim}(H) = f_n(k)$.

Proof. An easy trigonometric calculation shows that

$$\mathrm{perim}(H) = (n-k)\cdot 2\cos\frac{\pi}{n}\sin\frac{\pi}{n} + k\cdot 2\sin\frac{\pi}{n} = 2n\sin\frac{\pi}{n}\cos\frac{\pi}{n} + 2k\sin\frac{\pi}{n}\left(1-\cos\frac{\pi}{n}\right) = f_n(k). \quad \square$$

Lemma 6. *Let λ be the (minimum) distance from the origin to the boundary of W_n. Then $\lambda > f_n(n)/2$.*

Proof. Observe that $\lambda = 4\cos\frac{\pi}{n}$ and that $f_n(n) = 2n\sin\frac{\pi}{n}$. For $n \geq 5$, we have

$$\lambda = 4\cos\frac{\pi}{n} \geq 4\cos\frac{\pi}{5} = 3.236\ldots > \pi \geq n\sin\frac{\pi}{n} = \frac{f_n(n)}{2}. \quad \square$$

The following lemma establishes the reduction:

Lemma 7. *There is a vertex cover of k vertices for G if and only if there is an intersecting polygon of perimeter $f_n(k)$ for $\mathcal{Z}$.*

Proof. We first prove the direct implication. Suppose there is a vertex cover S of k vertices for G. We will find an intersecting polygon P of perimeter $f_n(k)$ for $\mathcal{Z}$. For each vertex i in S, select the corresponding vertex v_i of V_n. Let P be the convex hull of the n vertices of U_n and the k selected vertices of V_n. Then P contains at least one point from each polygonal chain in $\mathcal{Z}$. By Lemma 5, the perimeter of P is exactly $f_n(k)$.

We next prove the reverse implication. Suppose there is an intersecting polygon of perimeter $f_n(k)$ for $\mathcal{Z}$. We will find a vertex cover of k vertices for G. Let P be a minimum-perimeter intersecting polygon of $\mathcal{Z}$. Then P must be convex because otherwise the convex hull of P would be an intersecting polygon of even smaller perimeter. Since the n vertices of U_n are included in $\mathcal{Z}$ as n degenerate polygonal chains, the convex polygon P must contain U_n. It follows that P also contains the origin. Since $\mathrm{perim}(P) \leq f_n(k) < f_n(n)$, it follows by Lemma 6 that P cannot intersect the boundary of W_n, although it may intersect some segments $v_i w_i$. For each segment $v_i w_i$ that P intersects, the vertex v_i must be contained in P. This is because P is convex and contains the origin, while each segment $v_i w_i$ is on a line through the origin, and the origin is closer to v_i than w_i. Since P is a minimum-perimeter intersecting polygon, it must be the convex hull of the n vertices of U_n and some vertices of V_n. By Lemma 5, P contains at most k vertices of V_n, which, by construction, correspond to a vertex cover of at most k vertices of V. $\square$

It remains to make the reduction polynomial. We round each vertex coordinate in our construction to a rational number, and encode it in $O(z \cdot \log n)$ bits for a suitable large number z. Then each vertex is moved for a small distance at most $O(n^{-z})$ by the rounding. Recall the definition of $f_n(k)$ in (6), and observe that

$$2n\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \Theta(1), \quad 2\sin\frac{\pi}{n}\left(1-\cos\frac{\pi}{n}\right) = 2\sin\frac{\pi}{n}\cdot 2\sin^2\frac{\pi}{2n} = \Theta(n^{-3}). \tag{7}$$

In particular, $f_n(n) = \Theta(1)$ and $f_n(k+1/3) - f_n(k-1/3) = \Theta(n^{-3})$. Thus with a polynomial encoding size for each vertex, Lemma 6 still holds, and Lemma 5 can be relaxed to the following lemma:

Lemma 8. *Let H be the convex hull of the n vertices of U_n and any $k < n$ vertices of V_n. Then the perimeter of H is at least $f_n(k-1/3)$ and at most $f_n(k+1/3)$.*

Note that for any two different integers k_1 and k_2, the two ranges $[f_n(k_1 - 1/3), f_n(k_1 + 1/3)]$ and $[f_n(k_2 - 1/3), f_n(k_2 + 1/3)]$ are disjoint. Note also that for any integer $k < n$, $f_n(k + 1/3) < f_n(n)$. Consequently, Lemma 7 is relaxed to the following lemma, which preserves the reduction:

Lemma 9. *There is a vertex cover S of k vertices for G if and only if there is an intersecting polygon P of perimeter at least $f_n(k - 1/3)$ and at most $f_n(k + 1/3)$ for $\mathcal{Z}$.*

In a similar way, we can obtain a polynomial reduction in which the polygonal chains are slightly fattened into simple polygons with rational coordinates. This completes the proof of Theorem 3.

6 Concluding remarks

The problem MPIP is related to two other geometric optimization problems called largest and smallest convex hulls for imprecise points [10]. Given a set $\mathcal{R}$ of n regions that model n imprecise points in the plane, the problem largest (resp. smallest) convex hull is that of selecting one point from each region such that the convex hull of the resulting set P of n points is the largest (resp. smallest) with respect to area or perimeter. Note that MPIP is equivalent to smallest-perimeter convex hull for imprecise points as segments. The dual problem of largest convex hull for imprecise points as segments has been recently shown to be NP-hard [10] for both area and perimeter measures, and to admit a PTAS [9] for the area measure. We note that the core-set technique used in obtaining the PTAS for largest-area convex hull [9] cannot be used for MPIP because, for minimization, there could be many optimal or near-optimal solutions that are far from each other. For example, consider two long parallel segments that are very close to each other. Our Algorithm A2 overcomes this difficulty by Lemma 3.

The problem of computing a minimum-perimeter intersecting polygon of a set of simple polygons is also related to the traveling salesman problem with neighborhoods (TSPN), see [3, 5, 6]. Given a set $\mathcal{R}$ of n regions (neighborhoods) in the plane, TSPN is the problem of finding the shortest tour that visits at least one point from each neighborhood. Note that the shortest tour is the boundary of a (possibly degenerate) simple polygon but this polygon need not be convex. Although our NP-hardness reduction was established independently, it does have a certain similarity, as we noticed afterwards, to the APX-hardness reduction for TSPN in [5]. The convexity of an optimal solution in our case seems to limit this similarity, however, and prevented us from strengthening our NP-hardness result to an APX-hardness result.

We conclude with two open questions:

(1) Is the problem of computing a minimum-perimeter intersecting polygon of a set of segments NP-hard?

(2) Does the problem of computing a minimum-perimeter intersecting polygon of a set of simple polygons admit a PTAS or a constant-factor approximation algorithm?

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