Metric inequalities for polygons

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Abstract

Let $A_1, A_2, \ldots, A_n$ be the vertices of a polygon with unit perimeter, that is $\sum_{i=1}^{n} |A_i A_{i+1}| = 1$. We derive various tight estimates on the minimum and maximum values of the sum of pairwise distances, and respectively sum of pairwise squared distances among its vertices. In most cases such estimates on these sums in the literature were known only for convex polygons.

In the second part, we turn to a problem of Braß regarding the maximum perimeter of a simple n-gon (n odd) contained in a disk of unit radius. The problem was solved by Audet et al. [5], who gave an exact formula. Here we present an alternative simpler proof of this formula. We then examine what happens if the simplicity condition is dropped, and obtain an exact formula for the maximum perimeter in this case as well.

Keywords: Metric inequalities, polygon, perimeter, sum of distances.

1 Introduction

Let $A_1, A_2, \ldots, A_n$ be the vertices of a possibly self-crossing polygon (i.e., closed polygonal chain) with unit perimeter in the Euclidean plane. Here the perimeter is $\text{per}(A_1 A_2 \ldots A_n) = \sum_{i=1}^{n} |A_i A_{i+1}|$, where $A_{n+1} = A_1$. Let $s(n)$ be the infimum of the sum of pairwise distances among the n vertices, and $s_c(n)$ be the same infimum for the case of convex polygons:

$$s(n) = \inf_{\text{per}(A_1 A_2 \ldots A_n)=1} \sum_{i<j} |A_i A_j|. \tag{1}$$

$$s_c(n) = \inf_{\substack{\text{per}(A_1 A_2 \ldots A_n)=1 \\ A_1 A_2 \ldots A_n \text{ convex}}} \sum_{i<j} |A_i A_j|. \tag{2}$$

Larcher and Pillichshammer [14] proved that $s_c(n)$ grows linearly in n, and more precisely, that $s_c(n) \geq \frac{n-1}{2}$. Alternative proofs were recently given by Aggarwal [1] and Lükő [15]. We have nearly equality, if A_1 is close to $(0,0)$ and the other $n-1$ vertices A_i ($i > 1$) are all close to $(\frac{1}{2}, 0)$. Hence $s_c(n) = \frac{n-1}{2}$, as previously conjectured by Audet et al. [3]. Here we extend this result for arbitrary polygons and show that $s(n)$ has a similar behavior.

Theorem 1. *For every $n \geq 3$, $s(n) \geq \frac{n}{4}$. For n even equality holds; for n odd, $s(n) \leq \frac{n+1}{4}$.*

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Let now $S(n)$ be the supremum of the sum of pairwise distances among the vertices, and $S_c(n)$ be the same supremum for the case of convex polygons:

$$S(n) = \sup_{\mathrm{per}(A_1A_2\ldots A_n)=1} \sum_{i<j} |A_iA_j|. \tag{3}$$

$$S_c(n) = \sup_{\substack{\mathrm{per}(A_1A_2\ldots A_n)=1 \\ A_1A_2\ldots A_n \text{ convex}}} \sum_{i<j} |A_iA_j|. \tag{4}$$

Larcher and Pillichshammer [13] considered the following generalization of the sum of pairwise distances, for which they proved:

Theorem A. [13, Theorem 1] *Let* $f\colon [0,1/2] \to \mathbb{R}_0^+$ *be a function such that* $f(x)/x \leq 2f(1/2)$. *Then for any* $n \geq 3$ *and for any convex polygon with* n *vertices and unit perimeter we have*

$$\sum_{i<j} f(|A_iA_j|) \leq f\left(\frac{1}{2}\right) \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.$$

This bound is the best possible.

By taking $f(x) = x$, it follows from Theorem A (as proved in [13]) that $S_c(n)$ is quadratic in n, and more precisely, that $S_c(n) \leq \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, as previously conjectured by Audet et al. [3]. An alternative proof was given recently by Aggarwal [1] based on classical results of Altman [2] for convex polygons. We have nearly equality if $A_1, \ldots, A_{\lfloor n/2 \rfloor}$ are close to $(0,0)$ and $A_{\lfloor n/2 \rfloor+1}, \ldots, A_n$ are close to $(\frac{1}{2}, 0)$; see [3]. Hence the above upper bound is best possible, thus $S_c(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. Here we show that convexity can be dropped and the same inequality holds for arbitrary (not necessarily convex, and possibly self-crossing) polygons: that is, $S(n) \leq \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. This result has been also obtained recently by Lükő [15]. (Since both his proof as well as ours rely on the triangle inequality, both work in any metric space.)

Theorem 2. *For every* $n \geq 3$,

$$S(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.$$

Next we consider the sum of *squared* distances. Let now $t(n)$ be the infimum of the sum of pairwise squared distances among the vertices, and $t_c(n)$ be the same infimum for the case of convex polygons:

$$t(n) = \inf_{\mathrm{per}(A_1A_2\ldots A_n)=1} \sum_{i<j} |A_iA_j|^2. \tag{5}$$

$$t_c(n) = \inf_{\substack{\mathrm{per}(A_1,A_2,\ldots,A_n)=1 \\ A_1A_2\ldots A_n \text{ convex}}} \sum_{i<j} |A_iA_j|^2. \tag{6}$$

For convex polygons, it is known that $t_c(n)$ is linear in n. The current best lower bound, $t_c(n) \geq \frac{2n}{3\pi^2}$, is due to Januszewski [11]. From the other direction, placing A_1 near $(0,0)$, A_2 near $(\frac{1}{2}, 0)$ and the other $n-2$ points near the midpoint of A_1A_2, all in convex position, shows that $t_c(n) \leq \frac{n}{8}$ [16].

For arbitrary polygons, it is easy to make a construction for which this sum converges to $1/4$ as n tends to infinity. For even n, place the odd vertices at $(0,0)$, and the even vertices at $(\frac{1}{n}, 0)$.

Then $Z = \sum_{i<j} |A_iA_j|^2 = \frac{n^2}{4} \cdot \frac{1}{n^2} = \frac{1}{4}$. For odd n, place the odd vertices at $(0,0)$, and the even vertices at $(\frac{1}{n-1}, 0)$. Then $Z = \frac{n^2-1}{4} \cdot \frac{1}{(n-1)^2} = \frac{1}{4} \cdot \frac{n+1}{n-1} \to \frac{1}{4}$. Here we obtain a lower bound that is off by a factor of 2 (in the limit).

Theorem 3. *For every $n \geq 3$,*

$$\frac{1}{8} \leq t(n) \leq \frac{1}{4} + o(1).$$

Finally, let $T(n)$ be the supremum of the sum of pairwise squared distances among the vertices, and $T_c(n)$ be the same supremum for the case of convex polygons:

$$T(n) = \sup_{\mathrm{per}(A_1A_2\ldots A_n)=1} \sum_{i<j} |A_iA_j|^2. \tag{7}$$

$$T_c(n) = \sup_{\substack{\mathrm{per}(A_1A_2\ldots A_n)=1 \\ A_1A_2\ldots A_n \text{ convex}}} \sum_{i<j} |A_iA_j|^2. \tag{8}$$

By taking $f(x) = x^2$, it follows from Theorem A (see [13]) that $T_c(n)$ is quadratic in n, and more precisely, that $T_c(n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. See also [14] for a simpler proof of a slightly weaker upper bound, $T_c(n) \leq n^2/16$. An easy construction [14] (mentioned earlier in connection to $S_c(n)$) with vertices $A_1, \ldots, A_{\lfloor n/2 \rfloor}$ near $(0,0)$ and $A_{\lfloor n/2 \rfloor+1}, \ldots, A_n$ near $(1/2, 0)$, all in convex position, shows that the inequality $T_c(n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$ is tight: $T_c(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$. Again, here we show that convexity can be dropped and the same inequality holds for arbitrary (not necessarily convex, and possibly self-crossing) polygons. That is, $T(n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, and we obtain:

Theorem 4. *For every $n \geq 3$,*

$$T(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.$$

Both Theorem 2 and Theorem 4 follow from the more general statement formulated in Theorem 5.

Theorem 5. *Let $f\colon [0, 1/2] \to \mathbb{R}_0^+$ be a function such that $f(x)/x \leq 2f(1/2)$. Then for any $n \geq 3$ and for any polygon with n vertices and unit perimeter we have*

$$\sum_{i<j} f(|A_iA_j|) \leq f\left(\frac{1}{2}\right) \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil.$$

This bound is the best possible.

In the second part of the paper we turn to the following problem [8, p. 437] posed by Braß: For $n \geq 5$ odd, what is the maximum perimeter of a simple n-gon (n odd) contained in a disk of unit radius? The problem can be traced back to the collection of open problems in [7] (Problem 4, p. 449). A first solution was found by Audet et al. [5] (Theorem 6 below). Subsequently, another solution that also works in the hyperbolic plane was offered by Lángi [12]. Here we give yet another alternative solution.

As noted in [7, 8], for even n, one can come arbitrarily close to the trivial upper bound $2n$ by a simple polygon whose sides go back and forth near a diameter of the disk, but for odd n this construction does not work. Let Ω be a disk of unit radius, and let

$$F(n) = \sup_{\substack{\{A_1, A_2, \ldots, A_n\} \subset \Omega \\ A_1A_2\ldots A_n \text{ simple}}} \mathrm{per}(A_1A_2 \ldots A_n). \tag{9}$$

So trivially, $F(n) = 2n$, for even n. Fortunately, an exact formula for $F(n)$ can be also determined for odd n:

Theorem 6. [5]. *For every $n \geq 3$ odd,*

$$F(n) = \frac{\sqrt{8(n-2)^2 - 2 + 2\sqrt{1+8(n-2)^2}} \cdot \left(\sqrt{1+8(n-2)^2} + 3\right)}{4(n-2)}. \tag{10}$$

A natural question is: What happens if the simplicity condition is dropped? As before (for simple polygons) for even n, one can come arbitrarily close to the trivial upper bound $2n$; however, for odd n, the construction described previously (with the sides which go back and forth near a diameter of the disk) still does not work. Let

$$G(n) = \sup_{\{A_1,A_2,\ldots,A_n\} \subset \Omega} \operatorname{per}(A_1A_2\ldots A_n). \tag{11}$$

Clearly, $G(n) \geq F(n)$ holds, so $G(n) = 2n$, for even n. For odd n, we determine an exact formula for $G(n)$ as well:

Theorem 7. *For every $n \geq 3$ odd,*

$$G(n) = 2n \cos \frac{\pi}{2n}. \tag{12}$$

Notation. Throughout the paper, let P be a polygon with n vertices and unit perimeter, and $V(P) = \{A_1, A_2, \ldots, A_n\}$ denote its vertex set. Let $\ell(s)$ denote the line containing a segment s. Let $x(p)$ and $y(p)$ stand for the x- and y-coordinates of a point p. For brevity we denote the set $\{1, 2, \ldots, n\}$ by $[n]$.

Related problems and results. Various extremal problems on the sum of distances and respectively squares of distances among n points in $\mathbb{R}^d$ have been raised over time. For instance, more than 30 years ago, Witsenhausen [22] has conjectured that the maximum sum of squared distances among n points in $\mathbb{R}^d$, whose pairwise distances are each at most 1 is maximized when the points are distributed as evenly as possible among the $d+1$ vertices of a regular simplex of edge-length 1. He also proved that this maximum is at most $\frac{d}{2(d+1)}n^2$, which verified the conjecture at least when n is a multiple of $d+1$. The conjecture has been proved for the plane by Pillichshammer [18], and subsequently in higher dimensions by Benassi and Malagoli [6]. See also [1, 3, 4, 5, 9, 10, 11, 16, 17, 18, 19] for related questions on the sum of pairwise distances. In the spirit of Theorems 6 and 7, a mathematical puzzle from Winkler's collection [21, p. 114] asks for the minimum area of a simple polygon with an odd number of sides, each of unit length.

2 Preliminaries

The following simple fact is needed in the proof of Theorems 5.

Lemma 1. *Given an arbitrary polygon $P = A_1A_2\ldots A_n$, let $Q = A_{i_1}A_{i_2}\ldots A_{i_k}$, where $i_1 < i_2 < \ldots < i_k$, $3 \leq k \leq n$, be a sub-polygon of it. Then* $\operatorname{per}(Q) \leq \operatorname{per}(P)$.

Proof. By the triangle inequality, for any $j \in [k]$, we have

$$|A_{i_j}A_{i_{j+1}}| \leq \sum_{r=i_j}^{i_{j+1}-1} |A_rA_{r+1}|.$$

Adding up the above inequality over all $j \in [k]$ yields $\mathrm{per}(Q) \leq \mathrm{per}(P)$, as required. $\square$

We also need the following extension of Lemma 1 in [13] to the non-convex case. Its proof remains the same, since it does not use convexity; see [13].

Lemma 2. *Let $f\colon [0,1/2] \to \mathbb{R}_0^+$ be a function such that $f(x)/x \leq 2f(1/2)$; and let $n \geq 3$. Then for any n-vertex polygon with side lengths $a_1, \ldots, a_n$ and perimeter at most one, i.e., $\sum_{i=1}^n a_i \leq 1$, we have*

$$\sum_{i=1}^{n} f(a_i) \leq 2f\left(\frac{1}{2}\right).$$

Proof of Theorem 5. (Sketch.) The proof method is identical to that employed by Larcher and Pillichshammer [14] for the convex case; we give a sketch for completeness, and we refer the reader to their paper for details.

If n is even, a set of $\binom{\lfloor n/2 \rfloor}{2}$ quadrilaterals $\{Q_{ij}\}$, each a subpolygon of P, and a set of $\lfloor n/2 \rfloor$ edges $\{E_i\}$ are defined [14] so that the edges of the quadrilaterals Q_{ij} and the edges E_i form a partition of the edge set $\{A_iA_j \mid i < j\}$ (each edge appears exactly once). If n is odd, a set of $\binom{\lfloor n/2 \rfloor}{2}$ quadrilaterals $\{Q_{ij}\}$ and a set of $\lfloor n/2 \rfloor$ triangles $\{R_i\}$ are defined [14] (these quadrilaterals and triangles are subpolygons of P), so that the edges of the quadrilaterals Q_{ij} and of the triangles R_i form a partition of the edge set $\{A_iA_j \mid i < j\}$ (each edge appears exactly once).

If n is even, Lemma 1 yields that $\mathrm{per}(Q_{ij}) \leq 1$, and obviously $|E_i| \leq 1/2$ holds. If n is odd, Lemma 1 yields that $\mathrm{per}(Q_{ij}) \leq 1$, and $\mathrm{per}(R_i) \leq 1$. In each case (n even or odd), by Lemma 2, one can now bound from above the sum $\sum_{i<j} f(|A_iA_j|)$ by the same expression, $f(1/2)\lfloor n/2 \rfloor \lceil n/2 \rceil$, as required.

From the other direction, we have nearly equality if $A_1, \ldots, A_{\lfloor n/2 \rfloor}$ are close to $(0,0)$ and $A_{\lfloor n/2 \rfloor+1}, \ldots, A_n$ are close to $(\frac{1}{2}, 0)$. $\square$

3 Sum of distances: proofs of Theorems 1 and 2

Proof of Theorem 1. Let $p \in V(P)$ be an arbitrary vertex of P, and let $Z(p) = \sum_{q \in V(P)} |pq|$ be the sum of distances from p to the other vertices. The sum of pairwise distances Z satisfies $2Z = \sum_{p \in V(P)} Z(p)$. By the triangle inequality, for any $i \in [n]$, we have

$$|pA_i| + |pA_{i+1}| \geq |A_iA_{i+1}|.$$

By summing over $i \in [n]$, we get

$$2Z(p) \geq \sum_{i=1}^{n} |A_iA_{i+1}| = 1.$$

By summing over $p \in V(P)$, we get $4Z \geq n$, or $Z \geq n/4$, as required.

To see that this inequality is almost tight, construct a non-convex polygon as follows: For even n, place the odd vertices at $(0,0)$, and the even vertices at $(\frac{1}{n}, 0)$. Then $Z = \frac{n^2}{4} \cdot \frac{1}{n} = \frac{n}{4}$. For odd n, place the odd vertices at $(0,0)$, and the even vertices at $(\frac{1}{n-1}, 0)$. Then $Z = \frac{n^2-1}{4} \cdot \frac{1}{n-1} = \frac{n+1}{4}$. $\square$

Proof of Theorem 2. By taking $f(x) = x$, it follows from Theorem 5 that $S(n) \leq \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, as required. From the other direction, we clearly have $S(n) \geq S_c(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, and the equality $S(n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$ is proved. $\square$

4 Sum of squared distances: proofs of Theorems 3 and 4

We will need the following simple inequality:

Lemma 3. *Let AB be a segment of length a, and O be any point in the plane. Then*

$$|OA|^2 + |OB|^2 \geq \frac{a^2}{2} + 2y^2,$$

where y is the distance from O to the line $\ell(AB)$ determined by A and B. In particular, $|OA|^2 + |OB|^2 \geq a^2/2$.

Proof. Let M be the projection of O onto $\ell(AB)$, and write $a_1 = |MA|$, and $a_2 = |MB|$. Clearly,

$$|OA|^2 + |OB|^2 = a_1^2 + y^2 + a_2^2 + y^2 \geq \frac{a^2}{2} + 2y^2,$$

as desired. $\square$

Proof of Theorem 3. The proof follows the same line of argument as the proof of Theorem 1. Let $p \in V(P)$ be an arbitrary vertex of P, and let $Z(p) = \sum_{q \in V(P)} |pq|^2$ be the sum of squared distances from p to the other vertices. The sum of squared pairwise distances Z satisfies $2Z = \sum_{p \in V(P)} Z(p)$. By Lemma 3, for any $i \in [n]$, we have

$$|pA_i|^2 + |pA_{i+1}|^2 \geq \frac{|A_i A_{i+1}|^2}{2}.$$

By summing over $i \in [n]$, we get

$$2Z(p) \geq \frac{1}{2} \sum_{i=1}^{n} |A_i A_{i+1}|^2.$$

The Cauchy-Schwarz inequality yields

$$\sum_{i=1}^{n} |A_i A_{i+1}|^2 \geq \frac{1}{n} \left(\sum_{i=1}^{n} |A_i A_{i+1}| \right)^2 = \frac{1}{n}.$$

Hence $4Z(p) \geq \frac{1}{n}$ for any $p \in V(P)$. Summing up this inequality over all $p \in V(P)$ yields

$$8Z = 4 \sum_{p \in V(P)} Z(p) \geq \sum_{i=1}^{n} \frac{1}{n} = 1,$$

or $Z \geq 1/8$, as required. $\square$

Remark. Interestingly enough, besides the construction mentioned in the Introduction (with the odd vertices near $(0,0)$ and the even vertices near $(\frac{1}{n}, 0)$ or $(\frac{1}{n-1}, 0)$ depending on whether n is even or odd) there is yet another construction for which the sum of the squares of the distances is at most $1/4$ in the limit. For odd n, consider n points evenly distributed on a circle of radius $r = 1/(2n \cos \frac{\pi}{2n})$, and labeled from 1 to n, say in clockwise order. The polygon P is the thrackle which connects the point labeled i with the point labeled $i + \frac{n-1}{2}$ (as usually the indexes are taken modulo n). It is easy to verify that P has unit perimeter (see also Theorem 7). Write $Z = \sum_{i<j} |A_iA_j|^2$. We have

$$Z = n \sum_{i=1}^{(n-1)/2} 4r^2 \sin^2 \frac{i\pi}{n} = 4nr^2 \left(\sum_{i=1}^{(n-1)/2} \sin^2 \frac{i\pi}{n} \right).$$

Setting $k = \frac{n-1}{2}$ and $\alpha = \frac{\pi}{n}$ in the trigonometric identity [20, p. 64]

$$\sum_{i=1}^{k} \sin^2[i\alpha] = \frac{k+1}{2} - \frac{\sin[(k+1)\alpha] \cdot \cos[k\alpha]}{2\sin\alpha}$$

yields

$$\sum_{i=1}^{(n-1)/2} \sin^2 \frac{i\pi}{n} = \frac{n+1}{4} - \frac{1}{4} = \frac{n}{4} \implies Z = 4nr^2 \cdot \frac{n}{4} = \frac{n^2}{4n^2} \cdot \frac{1}{\cos^2 \frac{\pi}{2n}} = \frac{1}{4} \cdot \frac{1}{\cos^2 \frac{\pi}{2n}} \xrightarrow[n\to\infty]{} \frac{1}{4}.$$

For even n, duplicate one point in the construction above, and obtain a similar estimate.

Proof of Theorem 4. By taking $f(x) = x^2$, it follows from Theorem 5 that $T(n) \le \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, as required. From the other direction, we clearly have $T(n) \ge T_c(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$, and the proof of Theorem 4 is complete. ☐

5 Odd polygons: proofs of Theorems 6 and 7

Proof of Theorem 6. For $n = 3$ it is easily seen that the extremal polygon is an equilateral triangle of side $\sqrt{3}$, so let $n \ge 5$. Put $a = a(n) = \sqrt{1 + 8(n-2)^2}$ and let $H(n)$ be the right hand side of (10). Then $H(n)$ can be also written as

$$H(n) = \frac{[(a+1)^2 - 4]^{1/2}(a+3)}{4(n-2)}. \tag{13}$$

Clearly, we have $a \ge 2\sqrt{2}(n-2) \ge 2\sqrt{2} \ge 3/2$, hence $(a+1)^2 - 4 = a^2 + 2a - 3 \ge a^2$, and consequently $[(a+1)^2 - 4]^{1/2} \ge a$. We show that this implies the inequality $H(n) \ge 2(n-2) + \frac{3\sqrt{2}}{2}$:

$$H(n) \ge \frac{a(a+3)}{4(n-2)} \ge \frac{2\sqrt{2}(n-2)}{4(n-2)} \left(2\sqrt{2}(n-2) + 3\right) = \frac{2\sqrt{2}(n-2) + 3}{\sqrt{2}} = 2(n-2) + \frac{3\sqrt{2}}{2}. \tag{14}$$

Let $o = (0,0)$ be the center of Ω and X denote the horizontal diameter of Ω. Let P be an extremal (limit) polygon. Note that P may have overlapping edges but is otherwise non-crossing. We will show that P is unique and $\text{per}(P) = H(n)$. We start with the upper bound $\text{per}(P) \le H(n)$. Let BC be a longest side of P of length $|BC| = z \le 2$. We can assume that BC is horizontal. Label each side A_iA_{i+1} by 1 or 0 depending on whether it goes from left to right, or from right to left in the x-direction (vertical sides are labeled arbitrarily). Since n is odd, we can find two consecutive sides of P with the same label, say 1: they form a (weakly) x-monotone path, σ, of two edges. By relabeling the vertices (if necessary), we can assume that this path consists of the edges A_1A_2 and A_2A_3: $\sigma = A_1A_2A_3$. Let $L_1 = |\sigma| = |A_1A_2| + |A_2A_3|$, and let L_2 be the total edge length of the other $n-2$ edges, so that $\text{per}(P) = L_1 + L_2 \le |\sigma| + (n-2)z$.

Since $z \leq 2$ we can write $z = 2\sin\alpha$, for some $\alpha \in [0, \pi/2]$. We first note that if $z \leq \sqrt{3}$ the upper bound $\text{per}(P) \leq H(n)$ follows immediately. Indeed: (i) for $n = 5$, $\text{per}(P) \leq 5 \cdot \sqrt{3} = 8.66\ldots \leq H(5) = 8.97\ldots$ and we are done; (ii) for $n = 7$, $\text{per}(P) \leq 7 \cdot \sqrt{3} = 12.12\ldots \leq H(7) = 12.92\ldots$ and we are done; (iii) for $n \geq 9$, by (14), $\text{per}(P) \leq n\sqrt{3} \leq 2(n-2) + \frac{3\sqrt{2}}{2} \leq H(n)$, and we are also done. Therefore we can assume that $z \geq \sqrt{3} = 2\sqrt{3}/2$, hence $z = 2\sin\alpha$, for some $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$.

Lemma 4. *If* $z = 2\sin\alpha$, *for some* $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$, *then* $L_1 \leq 4\cos\frac{\alpha}{2}$.

Proof. Since the path $\sigma = A_1A_2A_3$ is x-monotone and the polygon P is non-crossing, vertical rays from interior points of BC meet σ on the same side of BC, if at all. Assume without loss of generality that σ lies above BC in this sense; see Fig. 1(left and center) for two examples. We may also assume that BC lies below o (i.e., $y(B) \leq 0$), since if there is a counterexample to the lemma with BC above o then there is also one with BC below o: translate BC down by $2y(B)$ (to a parallel position below o); observe that BC is still a horizontal segment of length z contained in Ω, and σ lies above BC.

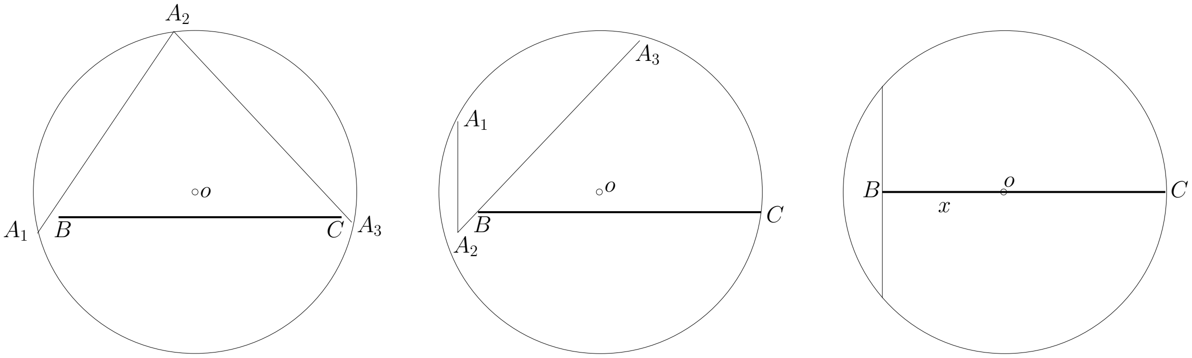

Figure 1: For a fixed length z, L_1 is maximum when BC is a chord of Ω.

Consider for a moment the case when BC is a right sub-segment of X as in Fig. 1(right). Let v be the length of the vertical chord incident to B. We have

$$v = 2\sqrt{1-x^2} = 2\sqrt{1-(z-1)^2} = 2\sqrt{1-(2\sin\alpha-1)^2} = 4\sqrt{\sin\alpha - \sin^2\alpha}.$$

We next verify that for $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$ we have

$$z + v < 4\cos\frac{\alpha}{2}, \tag{15}$$

or equivalently,

$$\sin\alpha + 2\sqrt{\sin\alpha - \sin^2\alpha} < 2\cos\frac{\alpha}{2}. \tag{16}$$

Observe that both $f(\alpha) = \sin\alpha + 2\sqrt{\sin\alpha - \sin^2\alpha}$ and $g(\alpha) = 2\cos\frac{\alpha}{2}$ are decreasing functions on the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$. Partition the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$ into two interior-disjoint intervals:

$$\left[\frac{\pi}{3}, \frac{\pi}{2}\right] = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2],$$

where $\alpha_1 = \pi/3$, $\beta_1 = \alpha_2 = 5\pi/12$, and $\beta_2 = \pi/2$. It is enough to check that $f(\alpha_i) < g(\beta_i)$, for $i = 1, 2$: $f(\alpha_1) = 1.547\ldots < g(\beta_1) = 1.586\ldots$, and $f(\alpha_2) = 1.328\ldots < g(\beta_2) = 1.414\ldots$. We have thereby verified (16).

According to whether the slopes of A_1A_2 and A_2A_3 are ≥ 0 or ≤ 0, we say that the path $\sigma = A_1A_2A_3$ is of type $++$, $+-$, $-+$, or $--$ (zero slopes are labeled arbitrarily). For example, σ in Fig. 1(left) is of type $+-$, while σ in Fig. 1(center) is of type $-+$. We distinguish three cases:

Case 1: The path $A_1A_2A_3$ is of type $-+$, as in Fig. 1(center) or in Fig. 2(left). If $x(A_2) \leq x(B)$ (the other case when $x(A_2) \geq x(C)$ is symmetric), then $|A_1A_2| \leq v$ and $|A_2A_3| \leq z$, and inequality (15) concludes the proof. Assume now that $x(B) < x(A_2) < x(C)$, thus A_2 lies above BC. The length of the chord extending BC is $2\sin\alpha_1$, for some α_1, where $\alpha \leq \alpha_1 \leq \pi/2$. By symmetry we can assume that $x(o) \leq x(A_2)$, as in Fig. 2. Let MN be the vertical chord through A_2, and $Q = BC \cap MN$. Assume that MN subtends a central angle 2β, for some $\beta \in [0, \pi/2]$. For fixed α and β, the length $|\sigma|$ is increased when A_2 is pushed down to Q, and A_3 is moved to M, i.e., $A_2 = Q$ and $A_3 = M$. The width and height of the rectangle with opposite vertices o and Q are $\cos\beta$ and $\cos\alpha_1 \leq \cos\alpha$, respectively. It follows that $|A_2A_3| \leq \cos\alpha + \sin\beta$. By the triangle inequality (used twice) we have

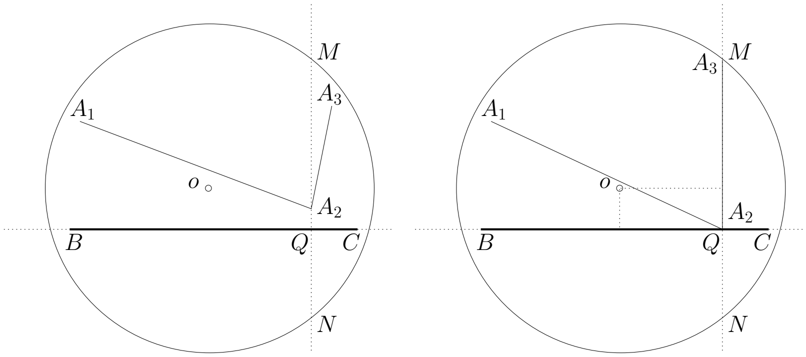

Figure 2: Case 1 in the proof of Lemma 4.

$$|A_1A_2| \leq |A_1o| + |oQ| \leq |A_1o| + \cos\alpha + \cos\beta \leq 1 + \cos\alpha + \cos\beta.$$

Recall the standard trigonometric inequality $\cos\beta + \sin\beta \leq \sqrt{2}$. Putting these together we obtain

$$\begin{aligned} |\sigma| &= |A_1A_2| + |A_2A_3| \leq (1 + \cos\alpha + \cos\beta) + (\cos\alpha + \sin\beta) \\ &= 1 + 2\cos\alpha + (\cos\beta + \sin\beta) \leq 1 + \sqrt{2} + 2\cos\alpha. \end{aligned} \tag{17}$$

Since $\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$, it remains to verify that $1 + \sqrt{2} + 4\cos^2\frac{\alpha}{2} - 2 \leq 4\cos\frac{\alpha}{2}$. Make the substitution $t = \cos\frac{\alpha}{2}$; then $t \in [\cos\frac{\pi}{4}, \cos\frac{\pi}{6}] = [\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}]$, and we need to verify that

$$1 + \sqrt{2} + 4t^2 - 2 \leq 4t,$$

or equivalently,

$$(2t-1)^2 \leq 2 - \sqrt{2}, \quad \text{for} \quad t \in [\sqrt{2}/2, \sqrt{3}/2]. \tag{18}$$

It is easy to see that for the above range of t we have

$$(2t-1)^2 \leq (2\sqrt{3}/2 - 1)^2 = (\sqrt{3}-1)^2 < 2 - \sqrt{2},$$

as required.

Case 2: The path $A_1A_2A_3$ is of type $+-$, as in Fig. 1(left). If $x(A_2) \leq x(B)$ (the other case when $x(A_2) \geq x(C)$ is symmetric), then $|A_1A_2| \leq v$ and $|A_2A_3| \leq z$, and inequality (15) concludes the proof. Otherwise, $x(B) < x(A_2) < x(C)$, and we replace $A_1A_2A_3$ by a longer path as follows. If the extension of A_2A_1 (beyond A_1) intersects BC, move A_1 to B; similarly if the extension of A_2A_3 (beyond A_3) intersects BC, move A_3 to C. Now σ is still x-monotone and the extensions of A_2A_1 and A_2A_3 meet $\partial\Omega$ without intersecting the interior of BC. Move A_1 and A_3 to these intersection points on the circle $\partial\Omega$. Now move A_2 upward to the circle $\partial\Omega$ while increasing $|\sigma|$. We now have a x-monotone path $A_1A_2A_3$ of type $+-$ and above BC with all three points A_1, A_2, A_3 on the circle.

Move BC downward until it hits $\partial\Omega$, and then rotate it around the endpoint on $\partial\Omega$; now BC is a chord of length z (subtending an angle of 2α from the center o) below the chord A_1A_3. Since A_1 and A_3 lie on the lower half-circle of $\partial\Omega$, the chord A_1A_3 subtends a central angle $2\alpha_1$, where $\alpha \le \alpha_1 \le \pi/2$. For a fixed α, $L_1 = |\sigma|$ is maximized when the triangle $\Delta A_1A_2A_3$ is isosceles with $|A_2A_1| = |A_2A_3|$, $|A_1A_3| = z = 2\sin\alpha$, and o is in the interior of the triangle. Indeed, $L_1 = 2(\sin\beta + \sin\gamma)$, where $2(\alpha_1 + \beta + \gamma) = 2\pi$, thus

$$L_1 = 2(\sin\beta + \sin\gamma) = 4\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2} \le 4\sin\frac{\beta+\gamma}{2} = 4\cos\frac{\alpha_1}{2} \le 4\cos\frac{\alpha}{2}.$$

Observe that in the (unique) maximizing position, $\beta = \gamma = \frac{\pi-\alpha}{2}$ and o is in the interior of the triangle $\Delta A_1A_2A_3$.

Case 3: The path $\sigma = A_1A_2A_3$ is of type $++$ (or symmetrically, $--$). Write $\delta = \angle A_1A_2A_3$. Since σ is also x-monotone, we have $\delta \ge \pi/2$. By the Cosine Theorem,

$$|A_1A_3|^2 = |A_1A_2|^2 + |A_2A_3|^2 - 2|A_1A_2||A_2A_3|\cos\delta \ge |A_1A_2|^2 + |A_2A_3|^2.$$

Obviously, $|A_1A_2|^2 + |A_2A_3|^2 \ge (|A_1A_2| + |A_2A_3|)^2/2$, hence

$$|\sigma| = |A_1A_2| + |A_2A_3| \le |A_1A_3|\sqrt{2} \le 2\sqrt{2} \le 4\cos\frac{\alpha}{2}, \quad \text{for} \quad \alpha \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right],$$

as required.

This concludes our case analysis. If any increase had occurred, a simple polygon whose perimeter is strictly larger than that of P could be constructed by taking the new path $A_1A_2A_3$ and then going back and forth near A_2A_3 with the remaining $n-2$ edges. However, this would contradict the fact that P were an extremal polygon. Observe that $L_1 \le 4\cos\frac{\alpha}{2}$ can hold with equality only in Case 2. This is clear for Case 1. Equality in Case 3 requires $\alpha = \pi/2$, thus $z = 2$; moreover, it requires $|\sigma| = 2\sqrt{2}$ and $|A_1A_2| = |A_2A_3| = \sqrt{2}$ with one of the two segments vertical and the other horizontal; however, the length of the vertical segment cannot exceed 1, which is a contradiction. We have thus shown that for a fixed length z, L_1 is maximized when BC (of length z) is a chord of Ω, $A_1 = B$, $A_3 = C$, and $|A_2A_1| = |A_2A_3|$ with A_1, A_2, A_3 on the circle and o in the interior of $\Delta A_1A_2A_3$. □

Since $z = 2\sin\alpha$ is the length of a longest side, by Lemma 4 we get

$$F(n) \le L_1 + (n-2)z \le 4\cos\frac{\alpha}{2} + 2(n-2)\sin\alpha. \tag{19}$$

We are thus led to maximizing the following function of one variable $\alpha \in [0, \pi/2]$:

$$f(\alpha) = 4\cos\frac{\alpha}{2} + 2(n-2)\sin\alpha.$$

The function $f(\cdot)$ is maximized at the root of the derivative:

$$f'(\alpha) = -2\sin\frac{\alpha}{2} + 2(n-2)\cos\alpha.$$

Making the substitution $x = \sin\frac{\alpha}{2}$, and using the trigonometric identity $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, yields the quadratic equation in x:

$$-2x + 2(n-2)(1-2x^2) = 0, \text{ or}$$

$$2(n-2)x^2 + x - (n-2) = 0.$$

The solution (corresponding to $\alpha \in [0, \pi/2]$) is

$$x = \sin\frac{\alpha}{2} = \frac{-1 + \sqrt{1 + 8(n-2)^2}}{4(n-2)}. \tag{20}$$

This implies

$$\cos\frac{\alpha}{2} = \sqrt{1 - \sin^2\frac{\alpha}{2}} = \frac{\sqrt{8(n-2)^2 - 2 + 2\sqrt{1+8(n-2)^2}}}{4(n-2)}.$$

Consequently, $F(n)$ is bounded from above by the maximum value of $f(\cdot)$, namely

$$\begin{aligned} F(n) &\leq 4\cos\frac{\alpha}{2} + 2(n-2)\sin\alpha = 4\cos\frac{\alpha}{2}\left((n-2)\sin\frac{\alpha}{2} + 1\right) \\ &= \frac{\sqrt{8(n-2)^2 - 2 + 2\sqrt{1+8(n-2)^2}} \cdot \left(\sqrt{1+8(n-2)^2} + 3\right)}{4(n-2)} = H(n). \end{aligned} \tag{21}$$

To see that this upper bound is tight construct a simple polygon as follows. Let A_1A_3 be a horizontal chord of length $z = 2\sin\alpha$, below the center o, with α set according to (20). Let A_2 be the intersection point above A_1A_3 of the vertical bisector of A_1A_3 with the unit circle $\partial\Omega$. The remaining $n-2$ sides of the polygon go back and forth near the horizontal chord A_1A_3. Thus formula (10) holds for every odd $n \geq 3$. This concludes the proof of Theorem 6. $\square$

Proof of Theorem 7. We start with the upper bound on $G(n)$. Consider the set of n-gons contained in Ω, where each such n-gon is given by the n-tuple of its vertex coordinates. Note that this forms a compact set, hence there exists an extremal polygon $P = A_1 \dots A_n$ (where $A_{n+1} = A_1$), which attains the maximum perimeter. Observe two properties of P that we justify below:

- Each vertex of P lies on $\partial\Omega$.
- All sides of P have equal length < 2.

First, assuming that A_i lies in the interior of Ω, per(P) could be increased by moving A_i orthogonally away from $A_{i-1}A_{i+1}$, or away from A_{i-1} in case $A_{i-1} = A_{i+1}$. This would contradict the maximality of P, hence all vertices of P lie on the circle. Second, assume now that A_{i-1}, A_i, A_{i+1} lie on the circle $\partial\Omega$, and $|A_{i-1}A_i| \neq |A_iA_{i+1}|$. Then per$(P)$ could be increased by moving A_i on the circle and further from $A_{i-1}A_{i+1}$ (to the midpoint of the arc). This again would contradict the maximality of P, hence all sides of P are equal. Since n is odd, it is obvious that the common edge length is strictly smaller than 2, since otherwise A_{n+1} cannot coincide with A_1.

Having established the two properties above, we can now easily obtain an upper bound on the perimeter of P. Let o be the center of Ω. For each $i \in [n]$, label the side A_iA_{i+1} by $+$ or $-$ depending on whether the center o lies on the right of $\overrightarrow{A_iA_{i+1}}$ or on the left of $\overrightarrow{A_iA_{i+1}}$. This labeling encodes the winding of the edges of P around the center o. Let $[n] = \Gamma_+ \cup \Gamma_-$ be the corresponding partition of $[n]$ determined by a positive or, respectively, negative labeling of A_iA_{i+1}. Write $k = |\Gamma_+|$ and $l = |\Gamma_-|$, where $k + l = n$.

We can assume that $l = 0$; indeed if both $k > 0$ and $l > 0$, then there exist two consecutive sides, $A_{i-1}A_i$ and A_iA_{i+1}, one with a positive label and one with a negative label. This implies that $A_{i-1} = A_{i+1}$, hence $\mathrm{per}(P)$ could be increased (recall that the side length is smaller than 2) by moving A_i to the point diametrically opposite to A_{i-1} (and A_{i+1}), a contradiction. Hence $l = 0$, $\Gamma_- = \emptyset$, and $\Gamma = \Gamma_+ = [n]$. For each $i \in [n]$, let $\angle A_i o A_{i+1} = 2\alpha$, where $0 < \alpha < \frac{\pi}{2}$. Since P is a closed polygonal chain, $2n\alpha = m\pi$ for some positive integer m, $1 \leq m \leq n-1$. Consequently, the perimeter of P is

$$\mathrm{per}(A_1 \dots A_n) = 2n \sin \alpha = 2n \sin \frac{m\pi}{2n} \leq 2n \sin \frac{(n-1)\pi}{2n} = 2n \cos \frac{\pi}{2n}, \tag{22}$$

as claimed.

It remains to show that this bound can be attained. Consider n points evenly distributed on the unit circle, and labeled from 1 to n, say in clockwise order. For the polygon consider the thrackle which connects the point labeled i with the point labeled $i + \frac{n-1}{2}$ (as usually the indexes are taken modulo n). It is easy to verify that its perimeter is given by the upper bound in (22), and this concludes the proof of Theorem 7. □

Remarks. For instance, $F(3) = 3\sqrt{3} = 5.19\dots$ corresponds to an equilateral triangle of side $\sqrt{3}$, and $F(5) = \sqrt{70 + 2\sqrt{73}} \cdot (\sqrt{73} + 3)/12 = 8.9774\dots$. The exact formula (10) easily yields an approximation of the form:

$$F(n) = 2(n-2) + 2\sqrt{2} + O\left(\frac{1}{n}\right).$$

Note that the sum of the first two terms in this formula, $2(n-2) + 2\sqrt{2}$, gives (in the limit) the perimeter of a simple polygon whose first two sides have length $\sqrt{2}$ each, and whose remaining $n-2$ sides go back and forth near a diameter of the unit disk. Thus the perimeter of the extremal polygon in Theorem 6 exceeds the perimeter of the polygon described above only by a term that tends to zero with n.

It is interesting to observe that (for odd n) in contrast to $F(n)$, $G(n)$ does get arbitrarily close to $2n$, as n tends to infinity; that is, $G(n) = 2n - o(1)$. Indeed, the series expansion of $\cos x$ around $x = 0$, $\cos x = 1 - \frac{x^2}{2} + \dots$ gives[1]

$$G(n) = 2n \cos \frac{\pi}{2n} = 2n \left(1 - \frac{\pi^2}{8n^2} + \dots\right) = 2n - \frac{\pi^2}{4n} + \dots = 2n - o(1).$$

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