

# The forest hiding problem\*

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## Abstract

Let  $\Omega$  be a disk of radius  $R$  in the plane. A set  $F$  of unit disks contained in  $\Omega$  forms a maximal packing if the unit disks are pairwise interior-disjoint and the set is maximal: i.e., it is not possible to add another disk to  $F$  while maintaining the packing property. A point  $p$  is hidden within the “forest” defined by  $F$  if any ray with apex  $p$  intersects some disk of  $F$ : that is, a person standing at  $p$  can hide without being seen from outside the forest. We show that if the radius  $R$  of  $\Omega$  is large enough, one can find a hidden point for any maximal packing of unit disks in  $\Omega$ . This proves a conjecture of Joseph Mitchell. We also present an  $O(n^{5/2} \log n)$ -time algorithm that, given a forest with  $n$  (not necessarily congruent) disks, computes the boundary illumination map of all disks in the forest.

**Keywords:** Illumination problem, maximal disk packing, blocking light rays.

## 1 Introduction

In this article we solve an illumination problem for maximal unit disk packings. A *unit* disk is a disk of unit diameter. Let  $\Omega$  be a disk of radius  $R$  in the plane. A set  $F$  of unit disks in the plane contained in  $\Omega$  forms a *maximal packing* if the unit disks are pairwise interior-disjoint and the set is maximal: i.e., it is not possible to add another disk to  $F$  while maintaining the packing property. A finite packing of (arbitrary radii) disks is referred to as a *forest*. A maximal packing of unit disks in  $\Omega$  is referred to as a *dense forest*. See Fig. 1. A point  $p$  is *dark* within the forest defined by  $F$  if any ray with apex  $p$  intersects some disk of  $F$ : i.e., a person standing at  $p$  cannot see out of the forest, or equivalently, he/she cannot be seen from outside the forest. A dark point is also referred to as a *hidden point*. A point that is not dark is *illuminated*. Consider any dense forest (with congruent disks) contained in  $\Omega$ . We show that if the radius  $R$  of the forest is large enough, then there exists a dark point in  $\Omega$  on some unit disk boundary. This answers an inspiring open problem posed by Joseph Mitchell at the 2007 Fall Workshop in Computational Geometry [10].

**Theorem 1.** *Any dense (circular) forest with congruent trees (of unit diameter) that is deep enough has a hidden point. More precisely, if the forest radius  $R$  is large enough, namely  $R \geq 2 \cdot 10^{108}$ , then there exist hidden points on the boundaries of at least  $R^2/14$  disks.*

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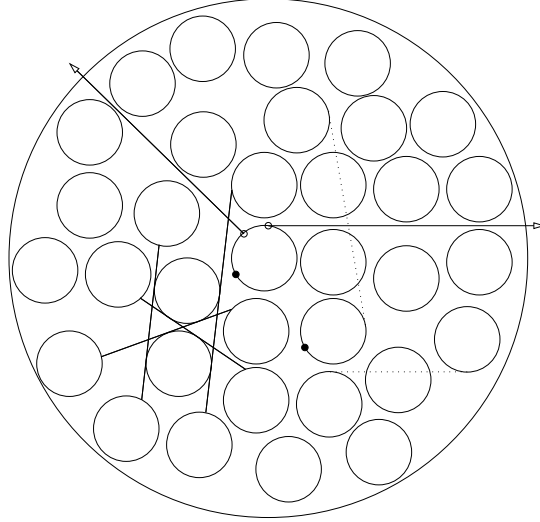


Figure 1: A dense forest with congruent trees. Two dark points (drawn as filled circles) and two illuminated points (drawn as empty circles) on some disk boundaries are shown. The four common tangents to a pair of disks are drawn with solid lines. Tangent segments (drawn with dotted lines) that intersect other disks are discarded by the algorithm for computing the boundary illumination map.

Let  $R^*$  be the supremum of such  $R$  for which a hidden point is not guaranteed to exist, i.e., there exists a forest of radius  $R$  so that all points on the boundaries or in between the disks are illuminated. Our Theorem 1 yields  $R^* \leq 2 \cdot 10^{108}$ , and we preferred simplicity to sharpness in our arguments. On the other hand, we would be surprised if  $R^* > 100$  were to hold.

It is interesting to note that with square trees, one can plant a forest that does not have a hidden point, no matter how deep and how dense it is; see Fig. 2. It is a good exercise for the reader to determine which part in the proof of Theorem 1 breaks down for this type of forest.

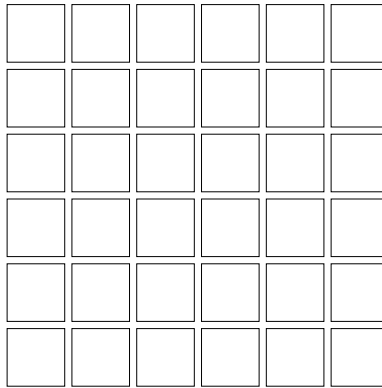


Figure 2: A deep and dense forest of square trees with no hidden point can be planted using this pattern.

In the final part of our paper we outline an algorithm that, given a forest with  $n$  (arbitrary) disks, determines for each disk boundary point, whether it is dark or illuminated. The *illumination map* of a forest  $F = \{\omega_1, \dots, \omega_n\}$  in  $\Omega$  is a binary function which specifies for each point in the free space among the trees,  $\Omega \setminus \bigcup_{i=1}^n \text{int}(\omega_i)$ , whether it is dark or illuminated. The *boundary illumination map* of a forest is the restriction of the above function to the boundaries of the disks

in  $F$ , that is, it specifies for each point  $p \in \partial\omega$ , where  $\omega \in F$ , whether it is dark or illuminated.

**Theorem 2.** *Given a (not necessarily dense) forest with  $n$  (not necessarily congruent) disks, there exists an  $O(n^{5/2} \log n)$ -time algorithm that computes the boundary illumination map of (all disks in) the forest.*

**Related problems and results.** Perhaps the closest related problem to the *forest hiding problem* we study here is the classical *orchard visibility problem* due to Polya [13]. In this problem, a tree is planted at each lattice point except the origin, in a circular orchard which has center at the origin and radius  $R$ . The question is how thick the trees must be in order to block completely the visibility from the origin. If the radius of the trees exceeds  $1/R$  units, one is unable to see out of the orchard in any direction from the origin. However, if the radii of the trees are smaller than  $1/\sqrt{R^2 + 1}$ , one can see out at certain angles. Various solutions due to Polya himself [13], Polya and Szego [14], Honsberger [5], Yaglom and Yaglom [15] have appeared over the years. Some new variants have been explored in [2, 6, 7, 9].

A problem “dual” to Mitchell’s *forest hiding problem*, where the trees are reflecting light rather than blocking it, is Pach’s *enchanted forest problem* with circular mirrors [3, p. 19], [11, p. 653]: If a match is lit in an enchanted forest of mirror trees, must some light escape to infinity? It is conjectured [12] that no finite collection of disjoint circular mirrors (i.e., a forest of mirror trees) can trap all the rays from a point light source. While Pach’s question can be asked for any forest, the answer is not known even for forests with congruent trees (i.e., unit disks).

## 2 Preliminaries

Let  $F$  be a dense forest of unit disks contained in  $\Omega$ . If a point on a unit disk boundary is illuminated by multiple light rays, we pick one such ray arbitrarily. In our proof, we will find hidden points on the boundaries of disks in  $F$ .

**Parameters.** We will use a set of carefully chosen and highly interdependent parameters, which we collect here for easy reference (recall that  $R$  is the forest radius):

$$\begin{aligned} \phi &= \pi/600, \quad \beta = \phi/2 = \pi/1200, \quad \theta = \pi/60. \\ \lambda &= 1/146000, \quad \lambda_1 = 1/200000, \quad \delta = \kappa = \lambda_1/3 = 1/600000. \\ R_2 &= 0.99R, \quad R_1 = R_2/\sqrt{2}, \quad L = 2R_1. \end{aligned} \tag{1}$$

**Notations and definitions.** Let  $\bar{r}$  be a light ray. Given two points  $s, t \in \bar{r}$ , where  $s$  is closer to the light source, for convenience we sometimes consider  $\bar{r}$  anchored at  $s$ , and sometimes at  $t$ . Fix an arbitrary coordinate system. The direction vector of a light ray is the unit vector along the ray from the light source. In the plane, we identify each direction vector with its angle  $\in [0, 2\pi)$ . Further we subdivide the set of all directions in 1200 *direction intervals* of angle  $\phi = 2\beta = 0.3^\circ$  each: set  $\beta = \pi/1200$ , and  $I(\alpha) = [\alpha - \beta, \alpha + \beta)$ ,  $\alpha = i\pi/600$ ,  $i = 0, 1, \dots, 1199$ . Thus each direction interval  $I(\alpha) = [\alpha - \beta, \alpha + \beta)$  is symmetric about its *main direction*  $\alpha$ . There are 1200 main directions of illumination and corresponding direction intervals symmetric about each of these main directions. Many of the arguments will be proven for  $\alpha = 3\pi/2$ , for convenience, and without any loss of generality. A ray  $\bar{r} \in I(3\pi/2)$  is said to be *almost vertical*.

Recall that  $\Omega$  is a disk of radius  $R$ . Let  $\Omega_1$  and  $\Omega_2$  be concentric disks of radii  $R_1$  and  $R_2$ , where  $R_2 = 0.99R$  and  $R_1 = R_2/\sqrt{2}$ . Thus  $R_1 < R_2 < R$ . Let  $L = 2R_1$ . For each of the main directions

$\alpha = i\pi/600$ ,  $i = 0, 1, \dots, 1199$ , let  $Q_\alpha$  be a square of side  $L$  and orientation  $\alpha$  concentric with  $\Omega$ , i.e., an axis-aligned square rotated counterclockwise by  $\alpha$ . Refer to Fig. 3 for a generic example ( $\alpha = 0$ ). By our setting of the parameters, we have

$$L = 2R_1 = \sqrt{2}R_2 = 0.99 \cdot \sqrt{2}R. \quad (2)$$

For any of the main directions  $\alpha$ , observe the inclusions:

$$\Omega_1 \subset Q_\alpha \subset \Omega_2 \subset \Omega. \quad (3)$$

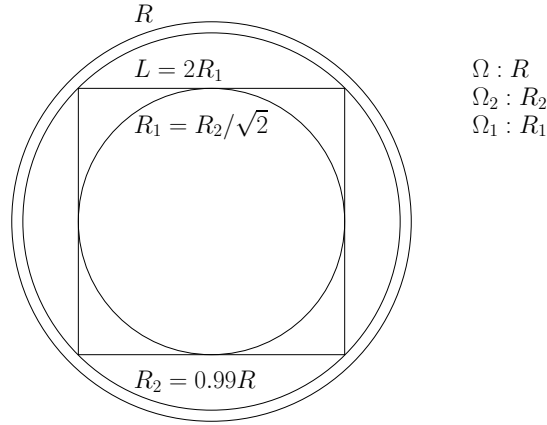


Figure 3: A generic example with  $\alpha = 0$ .

Let  $\gamma < 1/2$ . Given a segment  $s$ , the  $\gamma$ -central part of  $s$  is the smaller segment of length  $(1 - 2\gamma)|s|$  obtained from  $s$  by cutting off a small segment of length  $\gamma|s|$  from each end.

A disk in  $F$  is said to be *totally illuminated* if all its boundary points are illuminated by light rays. Otherwise the disk is said to be *partially illuminated*.

The following distinction among light rays illuminating disks is crucial in our arguments. Let  $\bar{r}$  be a light ray that illuminates a point on the boundary of a unit disk  $\omega$ . We say that  $\bar{r}$  *frontally illuminates*  $\omega$  if the extension of  $\bar{r}$  in the interior of the disk meets the  $\lambda$ -central part of the diameter of  $\omega$  orthogonal to  $\bar{r}$ ; otherwise we say that  $\bar{r}$  *tangentially illuminates*  $\omega$ . Correspondingly, we refer to the two types of rays as *frontal rays*, and respectively, *tangential rays*. See Fig. 4(left). A totally illuminated unit disk  $\omega$  is said to be *frontally illuminated* if at least one point on its boundary is frontally illuminated. A totally illuminated unit disk  $\omega$  is said to be *tangentially illuminated* if all points on its boundary are tangentially illuminated. Two important properties of these types of disk illumination are proved in Lemma 1.

**Lemma 1.** *Let  $\omega$  be a unit disk.*

- (i) *Let  $\bar{r}$  be a ray that tangentially illuminates  $\omega$ . Then the angle between  $\bar{r}$  and the tangent to  $\omega$  at the contact point  $p$  is at most  $\phi$ .*
- (ii) *Let  $\bar{r}$  be a ray that frontally illuminates  $\omega$ , with  $\bar{r}$  in the interval  $I(\alpha)$ , for some main direction  $\alpha$ . Then the extension of  $\bar{r}$  in the interior of the disk meets the  $\lambda_1$ -central part of the diameter of  $\omega$  orthogonal to direction  $\alpha$ .*

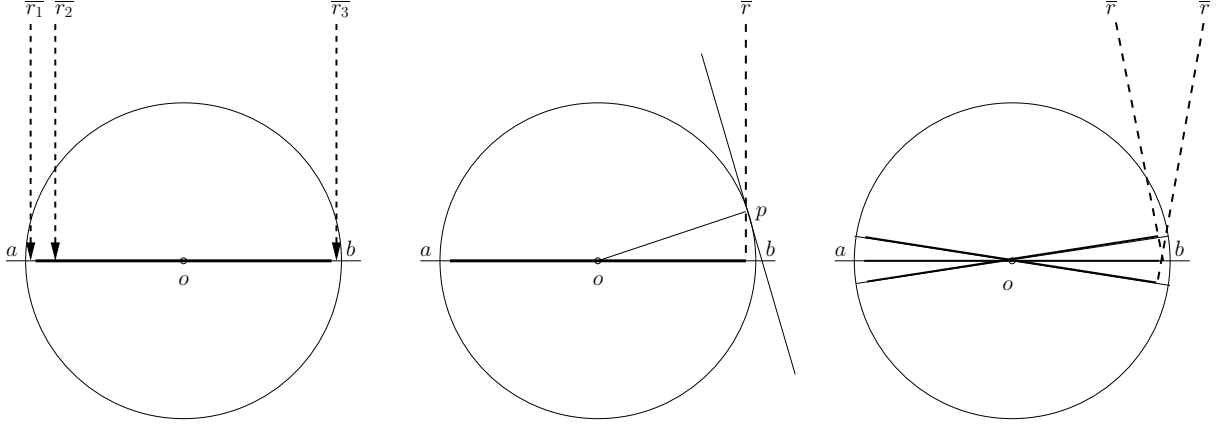


Figure 4: Left: two tangential vertical rays ( $\bar{r}_1$  and  $\bar{r}_3$ ) and a frontal vertical ray  $\bar{r}_2$  illuminating a unit disk from above. The central part of the horizontal diameter  $ab$  is drawn with a thicker line. Middle: A tangential ray must make an angle at most  $\phi$  with the tangent at the contact point  $p$ . Right: Two rays in the interval  $I(\alpha)$ , where  $\alpha = 3\pi/2$ , which frontally illuminate the disk from above.

**Proof.** Let  $ab$  be the horizontal diameter segment, and let  $o$  be the center of  $\omega$ .

(i) See Fig. 4(middle). Assume for convenience that the ray is vertical. By definition,  $\bar{r}$  meets  $ab$  at a point at distance at most  $\lambda$  from point  $b$ . Connect  $o$  with  $p$  and observe that the angle between  $\bar{r}$  and the tangent at  $p$  is the same as the angle  $\angle pob$ . We have

$$\angle pob \leq \arccos\left(\frac{\frac{1}{2} - \lambda}{\frac{1}{2}}\right) = \arccos(1 - 2\lambda) = \arccos\left(1 - \frac{1}{73000}\right) = 0.29990\dots^\circ < 0.3^\circ = \phi.$$

(ii) See Fig. 4(right). Assume for convenience that the main direction  $\alpha$  is vertical, i.e.,  $\alpha = 3\pi/2$ . By definition of direction intervals, the angle between  $\bar{r}$  and direction  $\alpha$  is at most  $\beta$ . It follows that the horizontal distance from the intersection of (the extension of)  $\bar{r}$  and  $ab$  to point  $b$  is at least

$$\frac{1}{2} - \frac{\frac{1}{2} - \lambda}{\cos \beta} = \frac{1}{2} - \frac{\frac{1}{2} - \frac{1}{146000}}{\cos \frac{\pi}{1200}} = 0.000005135\dots > \frac{1}{200000} = \lambda_1. \quad \square$$

We now establish two basic properties that follow from the maximality of the disk packing (Lemma 2 and Lemma 3).

**Lemma 2.** *Let  $U$  be a square of side 2 contained in  $\Omega$ . Then  $U$  contains the center of at least one disk in  $F$  in its interior.*

**Proof.** Denote by  $\xi$  the center of  $U$ . Assume for contradiction that the center of each disk in  $F$  intersecting  $U$  lies outside  $U$  or on  $\partial U$ . Then the open unit square  $V$  with center  $\xi$  and homothetic to  $U$  does not intersect any disk in  $F$ . Consequently, the unit disk  $\omega(\xi)$  centered at  $\xi$  is contained in  $V$  and is interior-disjoint from any unit disk in  $F$ , which in turn contradicts the maximality of the packing.  $\square$

**Lemma 3.** *Assume that  $R \geq 1000$ . Then there are at least  $3L^2/16$  disks from  $F$  in  $\Omega_1$ .*

**Proof.** The area of  $\Omega_1$  is  $\pi R_1^2$ . Superimpose an axis-aligned grid of squares of side 2 over  $\Omega_1$  with one of the squares centered at the center  $o$  of  $\Omega_1$ . The squares partition the area inside  $\Omega_1$ . Observe that the number of squares of side 2 that are contained in  $\Omega_1$  and are at distance at least  $1/2$  from its boundary equals the number of even lattice points (i.e., those with both coordinates even) contained in a circle of radius  $R_1 - \sqrt{2} - 1/2$  centered at  $o$ . This number equals the the number of lattice points contained in a circle of radius  $r = (R_1 - \sqrt{2} - 1/2)/2$  centered at  $o$ . By a well-known result of Gauss [4, p. 67], this number is at least  $\pi r^2 - 2\sqrt{2}\pi r$ . A straightforward calculation shows that for  $R \geq 1000$  we have

$$\pi r^2 - 2\sqrt{2}\pi r \geq 3.1r^2 \geq \frac{3R_1^2}{4} = \frac{3L^2}{16}.$$

By Lemma 2, every one of these squares contains the center of at least one unit disk in  $F$ , thus there are at least  $3L^2/16$  disks from  $F$  in  $\Omega_1$ .  $\square$

### 3 Proof of Theorem 1

**Proof outline.** In a nutshell the proof of Theorem 1 goes as follows. In Lemma 8 we prove that no disk in  $\Omega_1$  is tangentially illuminated. Assuming tangential illumination for such a disk would imply the existence a “hole” in the packing, a contradiction to its maximality. To make this verification easy, we have included a short computer program for computing intersection points between a suitable set of 7 rays almost tangent to a disk and 3 parallel lines. We defer the rather long proof of this technical lemma to the end of Section 3.

Second, we prove in Lemma 7 that the number of frontally illuminated disks in a relatively large disk  $\Omega_1$  (of radius  $R_1$ ) concentric with  $\Omega$  is at most a constant fraction (less than 1) of all the disks in  $\Omega_1$ . This part is the most difficult in the proof, relying on the sequence of lemmata 4 through 7. It then follows that the remaining disks in  $\Omega_1$  (a constant fraction of all disks) are only partially illuminated, therefore each has a hidden point on its boundary. We conclude that for large enough  $R$ , there exist many (that is,  $cR^2$  for an absolute constant  $c > 0$ ) distinct hidden points, each associated with a different disk in  $F$ . We now proceed with the details. Subsection 3.1, on frontal illumination is devoted to lemmata 4 through 7. Subsection 3.2, on tangential illumination is devoted to Lemma 8.

#### 3.1 Frontal illumination

We need the following weight distribution lemma. Some intuition is provided below. Given a (rectangular) region  $V$ , and a subset  $F' \subseteq F$  of the disk packing, the *weight* of  $V$  is the number of disk centers from  $F'$  that lie in  $V$ . Recall the setting of  $L$  in (1). Let  $Q$  be an axis-aligned square of side length  $L$  with weight  $W = \Theta(L^2)$ . An axis-aligned subrectangle  $V \subset Q$  of width  $L$  and height  $h$  is called a *separated heavy strip* in  $Q$  if: (i) the rectangle in  $Q \setminus V$  of width  $L$  bordering  $V$  from above has height  $h$ , and (ii) the weight of  $V$  is  $\Theta(L^2)$ .

Lemma 4 shows the existence of a separated heavy strip under a suitable assumption. The reason we want a separated heavy strip  $H$  is the following: Assuming that many disks,  $cL^2$ , from  $H$  are frontally illuminated from above (almost vertically), by the pigeonhole principle, we can find  $c'L$  rays that are anchored on the top side of  $Q$  in some short common interval  $K$  (for practical purposes,  $K$  acts as a single point). Now, a separated strip enforces a separated point set on the top

line of  $H$  (Lemma 5). In this context, a *separated point set* on the line has the minimum pairwise distance bounded from below by a positive constant.

**Lemma 4.** *Assume that  $R \geq 2^{16}$ . Let  $F' \subseteq F$  be a subset of at least  $W = L^2/8000$  disks. Let  $Q$  be an axis-aligned square of side length  $L$  containing all the disks in  $F'$ . Then there exists a separated heavy horizontal strip  $H \subset Q$  of height at least  $L/2^{15}$  that contains at least  $W/30 = L^2/240000$  disks in  $F'$ .*

**Proof.** Refer to Fig. 5 for a smaller example. We have  $W = L^2/8000$ . Since  $L \geq R$  (recall (1)), we also have  $L \geq 2^{16}$ . Consider a partition of  $Q$  into 16 horizontal strips of width  $L$ . The strips are labeled  $H_0$  through  $H_{15}$  in top to bottom order. The height of  $H_0$  is  $L/2^{15}$ . For  $i = 1, \dots, 15$ , the height of  $H_i$  is  $L/2^{16-i}$ .

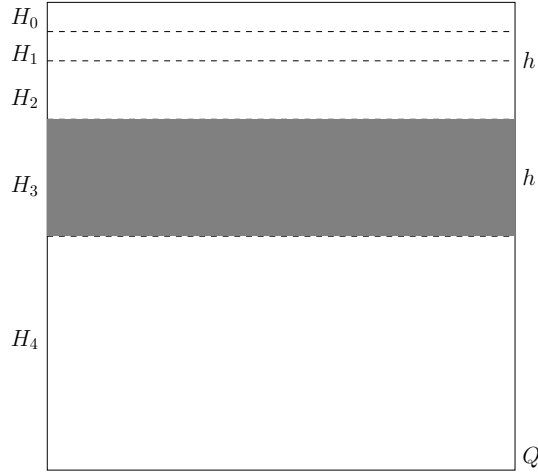


Figure 5: Partition of  $Q$  into 5 horizontal strips. Each of the strips  $H_1, H_2, H_3, H_4$  can be a separated heavy strip if its weight is  $\Theta(L^2)$ .

We distinguish two cases: If any of the strips  $H_1$  through  $H_{15}$  has weight at least  $W/30$ , we have found a separated heavy strip and we are done: (i) the strip has at least  $L^2/240000$  disk centers, and (ii) its height is  $L/2^{16-i} \geq L/2^{15}$ . So assume then that each of the 15 strips has weight smaller than  $W/30$ . Their total weight is less than  $15W/30 = W/2$ . It follows that the weight of  $H_0$  is at least  $W - W/2 = W/2 = L^2/16000$ . All disks whose centers lie in  $H_0$  are contained in rectangle  $H'_0 \supset H_0$  that extends  $H_0$  up and down by  $1/2$  (recall that disks could be “cut” only by horizontal strip boundaries). Thus  $H'_0$  contains at least  $L^2/16000$  disks. On the other hand, if  $L \geq 2^{16}$ , the height of  $H'_0$  is only

$$\frac{L}{2^{15}} + 1 \leq \frac{3L}{2^{16}}.$$

Since the disks form a packing, it follows by a simple area argument that the number of disks contained in  $H'_0$  is at most

$$L \left( \frac{L}{2^{15}} + 1 \right) \cdot \frac{4}{\pi} \leq \frac{3L^2}{2^{16}} \cdot \frac{4}{\pi} \leq \frac{L^2}{17000} < \frac{L^2}{16000}.$$

We have reached a contradiction, so the latter case cannot occur. □

A key step is the following lemma which shows a separation distance on  $\ell_2$  of the ray intersection points for almost vertical rays anchored in a small interval  $K$  on  $\ell_1$ .

**Lemma 5.** *Let  $\omega_1$  and  $\omega_2$  be two unit disks whose centers lie in a separated heavy horizontal strip  $H$  contained in an axis-aligned square  $Q$ . Let  $\ell_1$  and  $\ell_2$  be the horizontal lines through the top side of  $Q$  and the top side of  $H$ , respectively. Consider two almost vertical rays,  $\overline{r}_i$ ,  $i = 1, 2$ , so that  $\overline{r}_i$  frontally illuminates  $\omega_i$  from above, and  $\overline{r}_i$  is anchored at point  $s_i \in K$ , where  $K$  is a horizontal segment of length  $|K| = \kappa$  on  $\ell_1$ . If  $p_i$  denotes the intersection of the ray  $\overline{r}_i$  with the line  $\ell_2$ ,  $i = 1, 2$ , then  $|p_1 p_2| \geq \delta = 1/600000$ .*

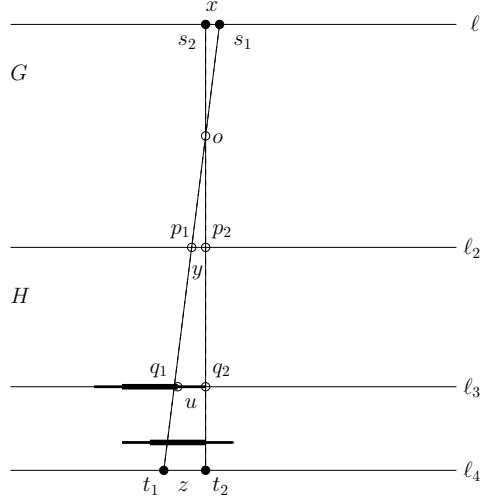


Figure 6: Illustration of Case 1 in the proof of Lemma 5.

**Proof.** Refer to Fig. 6. Denote by  $h$  the height of  $H$ . Let  $G$  be the horizontal strip of height  $h$  that separates  $H$  from the top side of  $Q$ . Assume without loss of generality that  $\omega_1$  is at least as high as  $\omega_2$  in  $H$ . Let  $\ell_3$  and  $\ell_4$  be the horizontal lines through the center of  $\omega_1$  and the bottom side of  $H$ , respectively. Let  $q_i$  be the intersection point of the ray  $\overline{r}_i$  with  $\ell_3$ , and let  $t_i$  be the intersection point of the extension of  $\overline{r}_i$  with  $\ell_4$ . Write  $x = |s_1 s_2|$ ,  $y = |p_1 p_2|$ ,  $u = |q_1 q_2|$ , and  $z = |t_1 t_2|$ . By Lemma 1(ii), since both disks are frontally illuminated by almost vertical rays, we have  $u \geq \lambda_1$ . We distinguish two cases:

*Case 1:*  $s_1 t_1$  and  $s_2 t_2$  cross inside the rectangle  $H \cup G$ . Denote by  $o$  the crossing point of  $s_1 t_1$  and  $s_2 t_2$ , and let  $h_i$  denote the vertical distance from  $o$  to  $\ell_i$ ,  $i = 1, 2, 3$ . First observe that  $o$  does not lie below  $\ell_3$ , since otherwise we would have by the similarity of the triangles  $\Delta o s_1 s_2$  and  $\Delta o q_1 q_2$

$$\frac{x}{u} = \frac{h_1}{h_3} = \frac{h + h_2}{h_3} \geq \frac{2h_2}{h_3} \geq \frac{2h_3}{h_3} = 2,$$

thus  $x \geq 2u \geq 2\lambda_1$ , contradicting the assumed inequality  $x \leq \kappa = \lambda_1/3$ .

We can therefore assume that the crossing point  $o$  lies above the horizontal line  $\ell_3$ . By the similarity of the triangles  $\Delta o s_1 s_2$  and  $\Delta o q_1 q_2$ , we have

$$\frac{h_1}{h_3} = \frac{x}{u} \leq \frac{\kappa}{\lambda_1} = \frac{1}{3}. \quad (4)$$

Hence  $h_1 \leq h_3/3 \leq 2h/3 < h$ , which means that  $o$  lies above  $\ell_2$  (i.e., in the strip  $G$ ); see Fig. 6.

By the similarity of the triangles  $\Delta os_1s_2$  and  $\Delta ot_1t_2$ , and respectively of the triangles  $\Delta op_1p_2$  and  $\Delta ot_1t_2$ , we have

$$\frac{h_1}{2h - h_1} = \frac{x}{z} \quad \text{and} \quad \frac{y}{z} = \frac{h - h_1}{2h - h_1}. \quad (5)$$

From the first equality it follows that

$$\frac{h_1}{h} = \frac{2x}{x + z}.$$

By plugging this value in the second equality, we get

$$y = z \cdot \frac{h - h_1}{2h - h_1} = z \cdot \frac{z - x}{2z} = \frac{z - x}{2} \geq \frac{u - x}{2} \geq \frac{\lambda_1 - \kappa}{2} = \kappa = \frac{1}{600000}.$$

*Case 2:*  $s_1t_1$  and  $s_2t_2$  do not cross inside  $H \cup G$ . First observe that  $s_1t_1$  and  $s_2t_2$  cannot be parallel, since then  $x = y = u \geq \lambda_1$ , which contradicts  $x \leq \kappa = \lambda_1/3$ . Similarly with previous arguments,  $s_1t_1$  and  $s_2t_2$  do not cross below  $\ell_4$ , since this would imply that  $x > u \geq \lambda_1$ , which contradicts  $x \leq \lambda_1/3$ . If  $s_1t_1$  and  $s_2t_2$  cross above  $\ell_1$ , then it is easy to see that  $y \geq u/2 \geq \lambda_1/2 > \kappa = 1/600000$ , as required.

This completes the proof of Lemma 5. □

We have shown that frontal illumination with rays in the same direction interval of a large set of disks contained in a separated heavy strip leads to a dense set of separated points on a line. We first establish an approximate equidistribution lemma for such sets (Lemma 6). This in turn will lead to a small set of almost parallel rays going inside the forest and creating a “hole” in the packing, again in contradiction to its maximality (Lemma 7, to follow).

**Lemma 6.** *Let  $k$  be a positive integer, and  $c, \delta > 0$  be two positive constants. Let  $I$  be an interval of length  $|I|$  containing at least  $c|I|$  points  $A = \{a_1, a_2, \dots\}$  at least  $\delta$  apart from each other. Put  $r = \frac{k}{k-1}$ ,  $j = \left\lceil \frac{\log \frac{2}{c\delta}}{\log r} \right\rceil$ , and set  $Z_0(k, c, \delta) = 2\delta \cdot k^j$ . Then if  $|I| \geq Z_0(k, c, \delta)$ , there exists a sub-interval  $J \subset I$  of length  $kx$ , for some  $x \geq 2\delta$ , such that in the uniform subdivision of  $J$  into  $k$  equal half-closed sub-intervals  $J_1, \dots, J_k$  of length  $x$ , each sub-interval contains a distinct point in  $A$ .*

**Proof.** Conduct an iterative process as follows. In step 0: Subdivide the interval  $I = I_0$  into  $k$  half-closed intervals<sup>1</sup> of equal length. If each of the  $k$  intervals contains at least one point, stop. Otherwise at least one of the  $k$  intervals is empty; now pick one of the remaining  $k - 1$  intervals, which contains the most points of  $A$ , say  $I_1$ . In step  $i$ ,  $i \geq 1$ : Subdivide  $I_i$  into  $k$  half-closed intervals of equal length and proceed as before.

In the current step  $i$ , the process either terminates successfully by finding an interval  $I_i$  subdivided into  $k$  sub-intervals, each containing at least one point in  $A$ , or it continues with another subdivision in step  $i + 1$ . We show that if  $|I|$  is large enough, and the number of subdivision steps is large enough, the iterative process terminates successfully.

Let  $L_0 = |I|$  be the initial interval length, and  $m_0 \geq c|I|$  be the (initial) number of points in  $I_0$ . At step  $i$ ,  $i \geq 0$ , let  $m_i$  be the number of points in  $I_i$ , and let  $L_i = |I_i|$  be the length of interval  $I_i$ . Clearly

$$L_i = \frac{|I|}{k^i}, \quad \text{and} \quad m_i \geq \frac{m_0}{(k-1)^i} \geq \frac{c|I|}{(k-1)^i}.$$

---

<sup>1</sup>When subdividing a closed interval, the first  $k - 1$  resulting sub-intervals are half-closed, and the  $k$ th sub-interval is closed. When subdividing a half-closed interval, all resulting sub-intervals are half-closed.

Let  $j$  be a positive integer so that

$$c \cdot \delta \cdot \left( \frac{k}{k-1} \right)^j \geq 2, \quad \text{e.g., set } j = \left\lceil \frac{\log \frac{2}{c\delta}}{\log r} \right\rceil, \quad \text{where } r = \frac{k}{k-1}.$$

Now set  $Z_0(k, c, \delta) = 2\delta \cdot k^j$ . If  $|I| \geq Z_0$ , as assumed, then by our choice of parameters we have

$$L_j = \frac{|I|}{k^j} \geq \frac{2\delta \cdot k^j}{k^j} = 2\delta, \quad (6)$$

and

$$m_j \cdot \delta \geq \frac{c|I| \cdot \delta}{(k-1)^j} = c \cdot \delta \cdot \left( \frac{k}{k-1} \right)^j \cdot \frac{|I|}{k^j} \geq \frac{2|I|}{k^j} = 2L_j. \quad (7)$$

Since the point set is  $\delta$ -separated, an interval packing argument on the line using (6) gives

$$m_j \cdot \delta \leq L_j + \frac{\delta}{2} + \frac{\delta}{2} = L_j + \delta \leq \frac{3}{2}L_j. \quad (8)$$

Observe that (8) is in contradiction to (7), which means that the iterative process cannot reach step  $j$ . We conclude that for some  $0 \leq i \leq j-1$ , step  $i$  is successful: we found an interval  $J := I_i$  of length  $|J| \geq 2k\delta$ , such that each of its  $k$  sub-intervals of length  $|J|/k \geq 2\delta$  contains at least one (distinct) point of  $A$ . This completes the proof.  $\square$

The large upper bound on the forest radius in our result is mainly due to the following lemma, which in turn is a consequence of the large numbers  $Z_0(k, c, \delta)$  in Lemma 6.

**Lemma 7.** *Assume that  $L \geq 380 \cdot Z_0(5, 1/(4.75 \cdot 10^{13}), 1/5) = 760 \cdot 5^{151} \approx 2.66 \cdot 10^{108}$ . For any direction interval  $I(\alpha)$ , at most  $L^2/8000$  disks in  $Q_\alpha$  are frontally illuminated by rays in the interval  $I(\alpha)$ . Consequently, at most  $3L^2/20$  disks in  $\Omega_1$  are frontally illuminated.*

**Proof.** It is enough to prove the first statement for a given  $\alpha$ , say  $\alpha = 270^\circ = 3\pi/2$ . Write  $Q = Q_{3\pi/2}$ , and recall that we refer to rays in the interval  $I(3\pi/2)$  as almost vertical rays. Let  $\ell_1$  be the horizontal line through the top side of  $Q$ . Assume for contradiction that at least  $W = L^2/8000$  disks in  $Q$  are frontally illuminated by almost vertical rays. By Lemma 4 with  $F'$  being this subset of disks, one can find a separated heavy strip  $H$  of height  $h$  containing at least  $W/30 = L^2/240000$  disks that are frontally illuminated by almost vertical rays (from above). Observe that all these rays are anchored on the top side of  $Q$  extended by at most  $L \tan \beta < L/200$  in each direction, thus in an interval of length at most  $1.01L$ . Recall that  $\kappa = 1/600000$ , and divide the extended top side of  $Q$  into at most  $1.01L/\kappa \leq 606000L$  intervals of length  $\kappa$ . By the pigeonhole principle, at least

$$\frac{L^2}{240000} \cdot \frac{1}{606000L} \geq \frac{L}{1.5 \cdot 10^{11}}$$

of these rays are anchored in some common interval  $K = [k_1 k_2] \subset \ell_1$  of length  $|K| = \kappa$ . Denote by  $\mathcal{R}$  this set of rays. Consider the horizontal line  $\ell_2$  through the top side of  $H$ . Observe that the rays in  $\mathcal{R}$  intersects  $\ell_2$  in an interval, say  $I$  of length at most

$$|I| \leq \kappa + 2 \cdot \frac{L}{2} \cdot \tan \beta \leq \kappa + \frac{L}{381} \leq \frac{L}{380},$$

since the vertical distance between  $\ell_1$  and  $\ell_2$  is at most  $L/2$ . Denote by  $A = a_0 < a_1 < \dots$  these intersection points on  $\ell_2$ . By Lemma 5, each pair of points  $a_i < a_j$  are separated by at least  $\delta = 1/600000$  on  $\ell_2$ . Select points  $a_{120000i}$ ,  $i = 0, 1, \dots$ , in total at least

$$\frac{1}{120000} \cdot \frac{L}{1.5 \cdot 10^{11}} = \frac{L}{1.8 \cdot 10^{16}}$$

points. The selected points have separation distance at least  $\frac{120000}{600000} = \frac{1}{5}$  on  $\ell_2$ . For convenience, still denote by  $A$ , and respectively  $\mathcal{R}$  this sparser subset of intersection points and corresponding rays. Observe that

$$\frac{380}{1.8 \cdot 10^{16}} \geq \frac{1}{4.75 \cdot 10^{13}}, \quad \text{and set } k = 5, \quad c = \frac{1}{4.75 \cdot 10^{13}}, \quad \delta = \frac{1}{5}.$$

The calculation in Lemma 6 with these parameters yields:

$$j = \left\lceil \frac{\log(4.75 \cdot 10^{14})}{\log(5/4)} \right\rceil = 152, \quad \text{and } 380 \cdot Z_0(5, 1/(4.75 \cdot 10^{13}), 1/5) = 760 \cdot 5^{151} \approx 2.66 \cdot 10^{108}.$$

By Lemma 6, if  $|I| \geq Z_0(k, c, \delta)$ , or equivalently,  $L \geq 380 \cdot Z_0(k, c, \delta)$ , there exists an interval  $J$  on  $\ell_2$  of length  $kx$ , for some  $x \geq 2/5$ , such that in the uniform subdivision of  $J$  into  $k = 5$  equal half-closed subintervals  $J_1, \dots, J_5$  of length  $x$ , each subinterval contains a distinct point in  $A$ . The  $i$ th interval is  $J_i = [j_i, j_{i+1})$ . Let  $a_1 < a_2 < a_3 < a_4 < a_5$  be these 5 points on  $\ell_2$ , and let  $s_i a_i$ ,  $i = 1, 2, 3, 4, 5$ , denote these 5 rays in  $\mathcal{R}$  ( $s_i$ ,  $i = 1, 2, 3, 4, 5$ , are the anchor points on  $\ell_1$  of these rays). See Fig. 7(left).

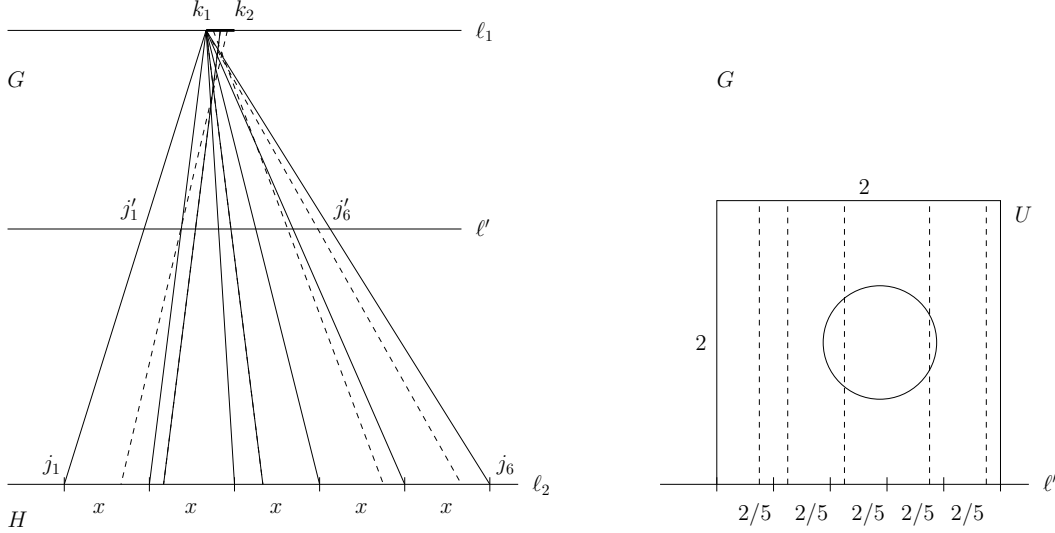


Figure 7: Five rays almost uniformly distributed, and a square of side 2 contained in  $\Omega$  and empty of unit disk centers. The light rays are drawn with dashed lines. The unit disk shown in the figure is infeasible.

Let  $\ell'$  be a horizontal line (parallel to  $\ell_2$ ) at vertical distance  $2h/5x$  from  $\ell_1$ . Let  $j'_1 < j'_2 < j'_3 < j'_4 < j'_5 < j'_6$  be the intersection points with  $\ell'$  of the six segments  $k_1 j_i$ . Observe that these points form a uniform subdivision of the interval  $[j'_1, j'_6]$  into 5 sub-intervals of length  $2/5$ . Observe that the segment  $k_1 a_i$  is a close approximation of the ray segment  $s_i a_i$ , where the latter can be obtained

from the former by keeping the common endpoint  $a_i$  fixed and moving the other endpoint on  $\ell_1$  by at most  $\kappa$ .

Consider now the axis-aligned square  $U$  of side 2 based on  $\ell'$ , with the two lower vertices at  $j'_1$  and  $j'_6$ . See Fig. 7(right). Note that  $U$  is contained in  $\Omega$ . We claim that  $U$  is empty of centers of disks in  $F$ . Let  $p \in U$  be any point in this square. Observe that the distance from  $p$  to one of the segments  $k_1 a_i$  is at most  $2/5 + 2 \tan \beta < 2/5 + 0.01 = 0.41$ , thus the distance from  $p$  to one of the ray segments  $s_i a_i$  is at most  $0.41 + \kappa < 0.42 < 1/2$ . This implies that a unit disk centered at  $p$  intersects at least one of the five rays  $s_i a_i$ . Thus  $p$  cannot be the center of a disk in  $F$ , and the emptiness claim for  $U$  follows, in contradiction to Lemma 2. We conclude that the initial assumption that at least  $L^2/8000$  disks in  $Q$  are frontally illuminated by almost vertical rays must be false. This proves the first statement of Lemma 7.

Recall (2) and (3). By summing over all 1200 direction intervals the above upper bound, we obtain that at most

$$1200 \cdot \frac{L^2}{8000} = \frac{3L^2}{20}$$

disks in  $\Omega_1$  are frontally illuminated. This completes the proof of Lemma 7.  $\square$

### 3.2 Tangential illumination

Note that if  $R \geq 100$ , then  $R - R_1 = R - 0.99R/\sqrt{2} \geq 20$ .

**Lemma 8.** *Assume that  $R \geq 100$ . Let  $\omega \in F$  be a unit disk whose center is at distance at least 20 from the boundary of  $\Omega$  (i.e., the center of  $\omega$  lies in a disk of radius  $R - 20$  concentric with  $\Omega$ ). Then  $\omega$  is not tangentially illuminated.*

**Proof.** Assume for contradiction that  $\omega$  is tangentially illuminated. Thus all its boundary points are tangentially illuminated. Consider a coordinate system in which the center of  $\omega$  is  $(-1/2, 0)$ . Let  $\theta = \pi/60$ , and  $\phi = \pi/600$ . Consider the following set of 7 points on the boundary of  $\omega$ ,  $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ , where

$$p_i = \frac{1}{2}(\cos(4 - i)\theta - 1, \sin(4 - i)\theta), \quad i = 1, 2, \dots, 7.$$

Observe that the 7 points are equidistributed on a circular arc of  $\omega$  subtending an angle of  $6\theta$ , and symmetrically placed about the middle point  $p_4 = (0, 0)$ , as in Fig. 8.

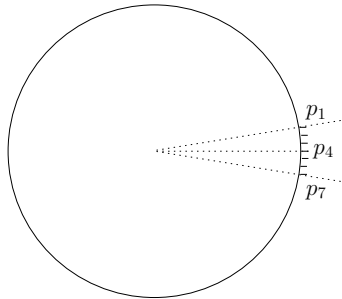


Figure 8: The set  $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$  on the boundary of  $\omega$ .

Each of these points must be illuminated by an almost tangent ray pointing up, or by an almost tangent ray pointing down, where for convenience we see the rays anchored at the illuminated boundary points. Let  $\bar{r}_i$  denote the ray illuminating  $p_i$ ,  $i = 1, 2, \dots, 7$ . By the pigeonhole principle, at least 4 rays are pointing in the “same” direction, and we can assume upwards, by symmetry. See also Fig. 9. Let  $\mathcal{R}_4$  denote the set of these 4 rays. Consider a generic horizontal line  $\ell = \ell(y)$ , where  $y \geq 3$ . The ray  $\bar{r}_i$  intersects  $\ell(y)$  at point  $(q_i, y)$ ,  $i = 1, 2, \dots, 7$ . For simplicity, we sometime omit the  $y$ -coordinate when it is not relevant, and refer to the point simply as  $q_i$ . The points  $(q_i, y)$ ,  $i = 1, 2, \dots, 7$ , on  $\ell(y)$  are ordered from left to right (this will be apparent below).

Consider one of the upward rays, say  $\bar{r}_i$ . The fact that  $\bar{r}_i$  is almost tangent to  $\omega$  at  $p_i$  implies that  $q_i \in [a_i, b_i]$ , where

1.  $(a_i, y)$  is intersection point of  $\ell(y)$  with the line tangent to  $\omega$  at  $p_i$ , whose angle is  $\pi/2 + (4-i)\theta$ .
2.  $(b_i, y)$  is the intersection point of  $\ell(y)$  with a line incident to  $p_i$  with angle  $\pi/2 + (4-i)\theta - \phi$ , i.e., obtained by rotating the tangent line clockwise by  $\phi$  around the (fixed) tangency point.

Observe that  $a_4 = 0$ . Note that  $a_i = a_i(y)$ ,  $b_i = b_i(y)$ , and  $q_i = q_i(y)$ , are linear functions of  $y$ . Further note, that for a fixed  $i < j$ , the differences  $b_j - a_i$  and  $a_j - b_i$  are increasing linearly with  $y$ . If  $\bar{r}_i$  is pointing up, color  $q_i$  red, otherwise color  $q_i$  blue; the blue points correspond to missing upwards rays. Out of the 7 points  $q_i$  on  $\ell$ , at least 4 are colored red.

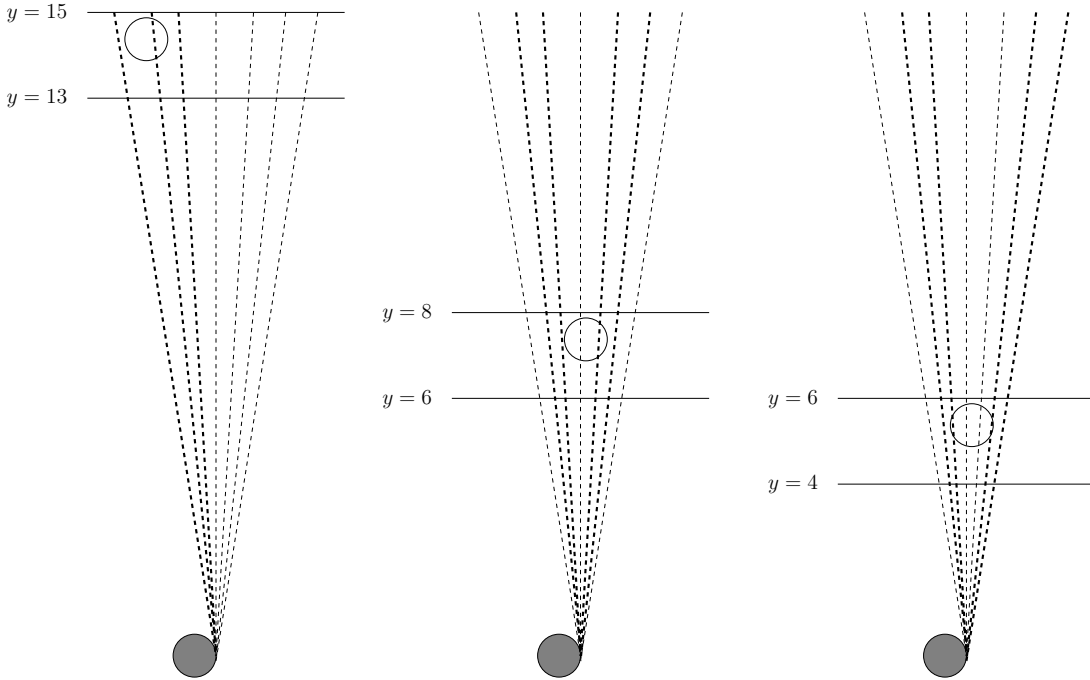


Figure 9: Case 1.1 (on the left), Case 2 (in the middle) and Case 3 (on the right). The 4 rays in  $\mathcal{R}_4$  are drawn with thicker dashed lines.

*Proof plan.* For each of the possibilities that can occur (in our case case analysis below) we exhibit a square  $U$  of side 2 contained in  $\Omega$ , that is empty of disk centers, and thus is in contradiction with Lemma 2.

Let the *red span* of  $\ell$  be the maximum difference  $j - i$ , where  $i < j$  and both  $q_i$  and  $q_j$  are red (this number is independent of  $y$ ). Let the *blue gap* of  $\ell$  be the maximum difference  $j - i$ ,

where  $i < j$ , both  $q_i$  and  $q_j$  are red, and all intermediate points (if any)  $q_{i+1}, \dots, q_{j-1}$  are blue (this number is also independent of  $y$ ). We distinguish three cases:

*Case 1:* There exist 3 consecutive monochromatic points  $q_l, q_{l+1}, q_{l+2}$  on  $\ell$ . Put  $r = l + 2$ . By symmetry with respect to the  $x$ -axis, we can assume they are red. Consider the two lines  $\ell(15)$  and  $\ell(13)$ . A straightforward calculation<sup>2</sup> gives the following values for  $a_i(y)$  and  $b_i(y)$ ,  $y = 15$ ,  $y = 13$ :

$$\begin{array}{llll}
y = 15 : & a_1 = -2.3695, & b_1 = -2.2895, & y = 13 : & a_1 = -2.0528, & b_1 = -1.9835, \\
& a_2 = -1.5738, & b_2 = -1.4947, & & a_2 = -1.3636, & b_2 = -1.2951, \\
& a_3 = -0.7854, & b_3 = -0.7068, & & a_3 = -0.6806, & b_3 = -0.6125, \\
& a_4 = 0.0000, & b_4 = 0.0785, & & a_4 = 0.0000, & b_4 = 0.0681, \\
& a_5 = 0.7868, & b_5 = 0.8657, & & a_5 = 0.6820, & b_5 = 0.7504, \\
& a_6 = 1.5793, & b_6 = 1.6590, & & a_6 = 1.3691, & b_6 = 1.4382, \\
& a_7 = 2.3820, & b_7 = 2.4630 & & a_7 = 2.0652, & b_7 = 2.1355
\end{array}$$

Observe that for  $i = 1, 2, 3, 4, 5, 6$ ,

$$b_{i+1}(15) - a_i(15) \leq b_7(15) - a_6(15) = 0.8837 < 1. \quad (9)$$

Also, for  $i = 1, 2, 3, 4, 5$ ,

$$a_{i+2}(13) - b_i(13) \geq a_5(13) - b_3(13) = 1.2945 > 1. \quad (10)$$

We distinguish 3 subcases, depending on whether the slopes of the rays through  $q_l$  and  $q_r$  are (1) both negative, (2) both positive, or (3) one negative and one positive; see Fig. 9. We use the convention that a vertical ray has positive slope.

*Case 1.1:*  $l = 1$ , and  $r = 3$ : the slopes of the rays through  $q_l$  and  $q_r$  are both negative. Consider the axis-aligned square  $U_1$  of side 2 centered at  $\xi_1 = ((q_l(13) + q_r(15))/2, 14)$ . We have

$$q_r(15) - q_l(13) \geq a_3(15) - b_1(13) = 1.1980 > 1.$$

By (9) and (10), the distance from any point  $u \in U_1$  to one of the three rays  $\overline{r_l}$ ,  $\overline{r_{l+1}}$ , or  $\overline{r_{l+2}}$  is less than  $1/2$ . It follows that  $U_1$  is empty of disk centers, in contradiction with Lemma 2.

*Case 1.2:*  $l = 2$ , or  $l = 3$ : the slope of the ray through  $q_l$  is negative, and the slope of the ray through  $q_r$  is positive. Consider the axis-aligned square  $U_2$  of side 2 centered at  $\xi_2 = ((q_l(13) + q_r(13))/2, 14)$ . We have

$$q_r(13) - q_l(13) \geq a_5(13) - b_3(13) = 1.2945 > 1.$$

It similarly follows that  $U_2$  is empty of disk centers, in contradiction with Lemma 2.

*Case 1.3:*  $l = 4$  or  $l = 5$ : the slopes of the rays through  $q_l$  and  $q_r$  are both positive. Consider the axis-aligned square  $U_3$  of side 2 centered at  $\xi_3 = ((q_l(15) + q_r(13))/2, 14)$ . We have

$$q_r(13) - q_l(15) \geq a_7(13) - b_5(15) = 1.1995 > 1.$$

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<sup>2</sup>The calculations for all three cases can be reproduced by using the C program listed in the Appendix.

It follows in the same way that  $U_3$  is empty of disk centers, in contradiction with Lemma 2.

For both *Case 2* and *Case 3*, there are no 3 consecutive monochromatic points on  $\ell$ . In particular this means that the red span is at least 4, and the blue gap is at most 3.

*Case 2:* There are no 3 consecutive monochromatic points on  $\ell$  and the red span is 4. Let  $l < r$  be the two indices with  $r - l = 4$  ( $q_l$  and  $q_r$  are the leftmost and respectively the rightmost points on  $\ell$ ). Observe that the ray through  $q_l$  has negative slope, and the ray through  $q_r$  has positive slope. Consider the two lines  $\ell(8)$  and  $\ell(6)$ . The corresponding values for  $a_i(y)$  and  $b_i(y)$ ,  $y = 8$ ,  $y = 6$  are:

$$\begin{array}{ll}
y = 8 : & a_1 = -1.2608, \quad b_1 = -1.2184, \\
& a_2 = -0.8381, \quad b_2 = -0.7960, \\
& a_3 = -0.4186, \quad b_3 = -0.3767, \\
& a_4 = 0.0000, \quad b_4 = 0.0419, \\
& a_5 = 0.4199, \quad b_5 = 0.4621, \\
& a_6 = 0.8436, \quad b_6 = 0.8862, \\
& a_7 = 1.2733, \quad b_7 = 1.3167 \\
y = 6 : & a_1 = -0.9441, \quad b_1 = -0.9123, \\
& a_2 = -0.6279, \quad b_2 = -0.5964, \\
& a_3 = -0.3138, \quad b_3 = -0.2824, \\
& a_4 = 0.0000, \quad b_4 = 0.0314, \\
& a_5 = 0.3151, \quad b_5 = 0.3468, \\
& a_6 = 0.6334, \quad b_6 = 0.6654, \\
& a_7 = 0.9565, \quad b_7 = 0.9892
\end{array}$$

Observe that for  $i = 1, 2, 3, 4, 5$ ,  $b_{i+2}(8) - a_i(8) \leq b_7(8) - a_5(8) = 0.8968 < 1$ . Also, for  $i = 1, 2, 3$ ,  $a_{i+4}(6) - b_i(6) \geq a_5(6) - b_1(6) = 1.2274 > 1$ . Consider the axis-aligned square  $U_4$  of side 2 centered at  $\xi_4 = ((q_l(6) + q_r(6))/2, 7)$ . We have

$$q_r(6) - q_l(6) \geq a_5(6) - b_1(6) = 1.2274 > 1.$$

It follows in the same way that  $U_4$  is empty of disk centers, in contradiction with Lemma 2.

*Case 3:* There are no 3 consecutive monochromatic points on  $\ell$  and the red span is at 5 or 6. Let  $l < r$  be the two corresponding indices, and observe again (as in the previous case) that the ray through  $q_l$  has negative slope, and the ray through  $q_r$  has negative slope. Consider the two lines  $\ell(6)$  and  $\ell(4)$ . The corresponding values for  $a_i(y)$  and  $b_i(y)$ ,  $y = 4$ , are:

$$\begin{array}{ll}
y = 4 : & a_1 = -0.6273, \quad b_1 = -0.6063, \\
& a_2 = -0.4177, \quad b_2 = -0.3968, \\
& a_3 = -0.2089, \quad b_3 = -0.1881, \\
& a_4 = 0.0000, \quad b_4 = 0.0209, \\
& a_5 = 0.2103, \quad b_5 = 0.2315, \\
& a_6 = 0.4232, \quad b_6 = 0.4446, \\
& a_7 = 0.6398, \quad b_7 = 0.6617
\end{array}$$

Observe that for  $i = 1, 2, 3, 4$ ,  $b_{i+3}(6) - a_i(6) \leq b_7(6) - a_4(6) = 0.9892 < 1$ . Also, for  $i = 1, 2$ ,  $a_{i+5}(4) - b_i(4) \geq a_6(4) - b_1(4) = 1.0294 > 1$ . Consider the axis-aligned square  $U_5$  of side 2 centered at  $\xi_5 = ((q_l(4) + q_r(4))/2, 5)$ . We have

$$q_r(4) - q_l(4) \geq a_6(4) - b_1(4) = 1.0294 > 1.$$

It follows as before that  $U_5$  is empty of disk centers, in contradiction with Lemma 2.

This completes the case analysis and thereby the proof of Lemma 8. □

### 3.3 Putting it all together: proof of Theorem 1

By Lemma 3 there are at least  $3L^2/16$  disks from  $F$  in  $\Omega_1$ . By Lemma 8, no disk in  $\Omega_1$  is tangentially illuminated, so by Lemma 7 the total number of disks in  $\Omega_1$  that are totally illuminated is at most  $3L^2/20$ . It follows that at least

$$\frac{3L^2}{16} - \frac{3L^2}{20} = \frac{3L^2}{80} \geq 1.96 \cdot \frac{3R^2}{80} \geq \frac{R^2}{14}$$

disks in  $\Omega_1$ , thus also in  $\Omega$ , are only partially illuminated, therefore each has a hidden point on its boundary. We conclude that for large enough  $R$ , there exist many distinct hidden points, each associated with a different disk in  $F$ . It is enough to choose  $R$  so that the condition in Lemma 7 is satisfied, namely

$$R \geq \frac{760 \cdot 5^{151}}{0.99\sqrt{2}} = 1.9018\dots \times 10^{108}.$$

Taking  $R \geq 2 \cdot 10^{108}$  will do, and the proof of Theorem 1 is complete.

**Discussion on parameters selection.** Most of the parameters are determined by  $\phi = \pi/600$  and  $\beta = \phi/2 = \pi/1200$ . The choice of the “wobble angle”  $\phi = \pi/600$  is needed in Lemma 8, and the value of the “classification angle”  $\beta$  chosen here is convenient for Lemma 1. The value  $\theta = \pi/60$  is one that works in Lemma 8.

## 4 Proof of Theorem 2

Let  $F = \{\omega_1, \dots, \omega_n\}$  be the set of  $n$  disks contained in  $\Omega$ . Each pair of disks in  $F$  admits four common tangent segments; see Fig. 1. Let  $\mathcal{T}$  denote the set of all  $O(n^2)$  tangent segments. If a tangent segment intersects any of the other  $n - 2$  disks, it is subsequently discarded (such as the rightmost tangent in Fig. 1). Let  $\mathcal{T}'$  denote the resulting set; clearly  $|\mathcal{T}'| = O(n^2)$ . For  $s \in \mathcal{T}'$ , let  $\ell(s)$  denote the supporting line of  $s$ . Each tangent segment in  $\mathcal{T}'$  is extended both ways until it hits one of the disks or to infinity (equivalently, until it hits the enclosing disk  $\Omega$ ). Since  $|\mathcal{T}'| = O(n^2)$ , we obtain a collection  $\mathcal{I}$  of  $O(n^2)$  circular *elementary* intervals on the boundaries of the disks, that is, intervals without any other tangent-with-disk intersection points in their interior. The sets  $\mathcal{T}'$  and  $\mathcal{I}$  can be computed in  $O(n^{5/2} \log n)$  time as follows. We use the ray shooting data structure of Agarwal et al. [1, Theorem 7.1], building on an earlier method of van Kreveld et al. [8].

The data structure  $\mathcal{D}$  stores a collection of  $n$  pairwise disjoint disks, for answering ray shooting queries: for a query ray  $\rho$ , the first disk hit by the ray is returned as the answer to the query.  $\mathcal{D}$  can be constructed in  $O(n^2 \log n)$  time and uses  $O(n \log n)$  space. The query processing time is  $O(\sqrt{n} \log n)$ . For each element  $s \in \mathcal{T}$ , three queries with rays collinear with  $\ell(s)$  are performed: the rays are anchored at the two endpoints of  $s$ . The first query determines if  $s$  is discarded, and if the answer is negative, i.e.,  $s \in \mathcal{T}'$ , the remaining two queries find the intersection points with the first disk hit or with  $\partial\Omega$ , in either direction. Since there are  $O(n^2)$  queries overall, they can be processed in  $O(n^{5/2} \log n)$  time. Once all queries have been answered, for each disk  $\omega \in F$ , the circular order of the intersection points on its boundary is computed to find the elementary intervals. This takes overall  $O(n^2 \log n)$  time (for all  $n$  disks). Observe that all points in an elementary interval are equivalent with respect to illumination. Note also that each of the  $O(n^2)$  endpoints of the elementary intervals determines a dark/illuminated labeling for the left and right adjacent intervals, as a function of the position of the tangent line relative to the two disks involved. This information can be used to determine the status (dark versus illuminated) of each interval in

$\mathcal{I}$  with no additional time overhead. The resulting time complexity of the algorithm is therefore dominated by the overall query processing time, hence it is  $O(n^{5/2} \log n)$ .

**Remark.** One can construct small examples of a dense forest with congruent (unit radius) disks with interior hidden points, but no boundary hidden points. See Fig. 10: the construction is symmetric with respect to the two coordinate axes, with the center of the rightmost disk at  $(\frac{16}{7}, 0)$ , and the center of the top disk at  $(0, \frac{17}{16})$ . Let  $\alpha$  be the angle of the tangent line (of positive slope) through the origin  $o = (0, 0)$  to the rightmost disk, as shown in the figure. Let  $\beta$  be the angle of the tangent line (of positive slope) through the origin  $o = (0, 0)$  to the top disk, as shown in the figure. By construction, we have  $\sin \alpha = \frac{7}{16}$ , and  $\cos \beta = \frac{16}{17}$ . To see that the origin  $o$  is dark, it is enough to check that  $\beta < \alpha$ , or equivalently, that  $\sin \beta < \sin \alpha$ :  $\frac{\sqrt{33}}{17} < \frac{7}{16}$ . Similarly, one can verify that all boundary points are illuminated, by computing the angles made by the two tangent lines through the point  $(0, \frac{1}{16})$  to the rightmost and to the lowest disk.

By Theorem 1, there are no such examples if the forest radius is large enough (i.e.,  $R \geq 2 \cdot 10^{108}$ ). However if one gives up the density constraint, the example in Fig. 10 can be extended for an arbitrary large number of congruent disks. This also brings up the question whether computing the (entire) illumination map of a forest is harder than computing the boundary illumination map. Note that computing the arrangement of tangent lines with the same idea as in the proof of Theorem 2 gives a polynomial time algorithm, however its time complexity is higher.

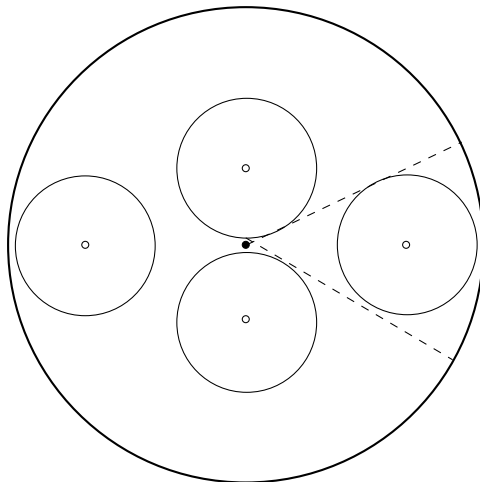


Figure 10: An interior hidden point (filled circle) in a dense forest with 4 congruent trees, and no boundary hidden point.

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## A Source code

```
#include <math.h>
#include <stdio.h>

#define Y15 0
#define Y13 1
#define Y8 2
#define Y6 3
#define Y4 4

double a[8][5], b[8][5];

double theta = M_PI / 60;
double phi = M_PI / 600;

void compute(double y, int Y) {
    int i;

    for (i = 1; i <= 7; i++) {
        double alpha[8], beta[8];

        alpha[i] = (4 - i) * theta;
        beta[i] = alpha[i] - phi;

        a[i][Y] = 0.5 * (cos(alpha[i]) - 1)
            - (y - 0.5 * sin(alpha[i])) * tan(alpha[i]);
        b[i][Y] = 0.5 * (cos(alpha[i]) - 1)
            - (y - 0.5 * sin(alpha[i])) * tan(beta[i]);

        printf("a%d(%g) = % 6.4f  b%d(%g) = % 6.4f\n",
            i, y, a[i][Y], i, y, b[i][Y]);
    }
    printf("\n");
}

int main() {
    int i;

    compute(15, Y15);
    compute(13, Y13);
    compute(8, Y8);
    compute(6, Y6);
    compute(4, Y4);

    printf("Case 1\n");
    for (i = 1; i <= 6; i++)
        printf("b%d(15) - a%d(15) = % 6.4f\n",
            i + 1, i, b[i + 1][Y15] - a[i][Y15]);
    printf("\n");
    for (i = 1; i <= 5; i++)
        printf("a%d(13) - b%d(13) = % 6.4f\n",
            i + 2, i, a[i + 2][Y13] - b[i][Y13]);
}
```

```

printf("\n");

printf("Case 1.1\n");
printf("a3(15) - b1(13) = % 6.4f\n", a[3][Y15] - b[1][Y13]);
printf("\n");

printf("Case 1.2\n");
printf("a4(13) - b2(13) = % 6.4f\n", a[4][Y13] - b[2][Y13]);
printf("a5(13) - b3(13) = % 6.4f\n", a[5][Y13] - b[3][Y13]);
printf("\n");

printf("Case 1.3\n");
printf("a6(13) - b4(15) = % 6.4f\n", a[6][Y13] - b[4][Y15]);
printf("a7(13) - b5(15) = % 6.4f\n", a[7][Y13] - b[5][Y15]);
printf("\n");

printf("Case 2\n");
for (i = 1; i <= 5; i++)
    printf("b%d(8) - a%d(8) = % 6.4f\n",
        i + 2, i, b[i + 2][Y8] - a[i][Y8]);
printf("\n");
for (i = 1; i <= 3; i++)
    printf("a%d(6) - b%d(6) = % 6.4f\n",
        i + 4, i, a[i + 4][Y6] - b[i][Y6]);
printf("\n");

printf("Case 3\n");
for (i = 1; i <= 4; i++)
    printf("b%d(6) - a%d(6) = % 6.4f\n",
        i + 3, i, b[i + 3][Y6] - a[i][Y6]);
printf("\n");
for (i = 1; i <= 2; i++)
    printf("a%d(4) - b%d(4) = % 6.4f\n",
        i + 5, i, a[i + 5][Y4] - b[i][Y4]);
return 0;
}

```

## B Output

```

a1(15) = -2.3695  b1(15) = -2.2895
a2(15) = -1.5738  b2(15) = -1.4947
a3(15) = -0.7854  b3(15) = -0.7068
a4(15) = -0.0000  b4(15) =  0.0785
a5(15) =  0.7868  b5(15) =  0.8657
a6(15) =  1.5793  b6(15) =  1.6590
a7(15) =  2.3820  b7(15) =  2.4630

```

```

a1(13) = -2.0528  b1(13) = -1.9835
a2(13) = -1.3636  b2(13) = -1.2951
a3(13) = -0.6806  b3(13) = -0.6125
a4(13) = -0.0000  b4(13) =  0.0681
a5(13) =  0.6820  b5(13) =  0.7504

```

a6(13) = 1.3691 b6(13) = 1.4382  
a7(13) = 2.0652 b7(13) = 2.1355

a1(8) = -1.2608 b1(8) = -1.2184  
a2(8) = -0.8381 b2(8) = -0.7960  
a3(8) = -0.4186 b3(8) = -0.3767  
a4(8) = -0.0000 b4(8) = 0.0419  
a5(8) = 0.4199 b5(8) = 0.4621  
a6(8) = 0.8436 b6(8) = 0.8862  
a7(8) = 1.2733 b7(8) = 1.3167

a1(6) = -0.9441 b1(6) = -0.9123  
a2(6) = -0.6279 b2(6) = -0.5964  
a3(6) = -0.3138 b3(6) = -0.2824  
a4(6) = -0.0000 b4(6) = 0.0314  
a5(6) = 0.3151 b5(6) = 0.3468  
a6(6) = 0.6334 b6(6) = 0.6654  
a7(6) = 0.9565 b7(6) = 0.9892

a1(4) = -0.6273 b1(4) = -0.6063  
a2(4) = -0.4177 b2(4) = -0.3968  
a3(4) = -0.2089 b3(4) = -0.1881  
a4(4) = -0.0000 b4(4) = 0.0209  
a5(4) = 0.2103 b5(4) = 0.2315  
a6(4) = 0.4232 b6(4) = 0.4446  
a7(4) = 0.6398 b7(4) = 0.6617

Case 1

b2(15) - a1(15) = 0.8748  
b3(15) - a2(15) = 0.8670  
b4(15) - a3(15) = 0.8640  
b5(15) - a4(15) = 0.8657  
b6(15) - a5(15) = 0.8722  
b7(15) - a6(15) = 0.8837

a3(13) - b1(13) = 1.3029  
a4(13) - b2(13) = 1.2951  
a5(13) - b3(13) = 1.2945  
a6(13) - b4(13) = 1.3010  
a7(13) - b5(13) = 1.3148

Case 1.1

a3(15) - b1(13) = 1.1980

Case 1.2

a4(13) - b2(13) = 1.2951  
a5(13) - b3(13) = 1.2945

Case 1.3

a6(13) - b4(15) = 1.2906  
a7(13) - b5(15) = 1.1995

Case 2

$$\begin{aligned}b_3(8) - a_1(8) &= 0.8841 \\b_4(8) - a_2(8) &= 0.8800 \\b_5(8) - a_3(8) &= 0.8807 \\b_6(8) - a_4(8) &= 0.8862 \\b_7(8) - a_5(8) &= 0.8968\end{aligned}$$

$$\begin{aligned}a_5(6) - b_1(6) &= 1.2274 \\a_6(6) - b_2(6) &= 1.2298 \\a_7(6) - b_3(6) &= 1.2389\end{aligned}$$

Case 3

$$\begin{aligned}b_4(6) - a_1(6) &= 0.9755 \\b_5(6) - a_2(6) &= 0.9747 \\b_6(6) - a_3(6) &= 0.9792 \\b_7(6) - a_4(6) &= 0.9892\end{aligned}$$

$$\begin{aligned}a_6(4) - b_1(4) &= 1.0294 \\a_7(4) - b_2(4) &= 1.0365\end{aligned}$$