A Problem on Track Runners

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Abstract

Consider the unit circle $C$ and a circular arc $A$ of length $\ell = |A| < 1$. It is shown that there exists $k = k(\ell) \in \mathbb{N}$, and a schedule for $k$ runners with $k$ distinct but constant speeds so that at any time $t \geq 0$, at least one of the $k$ runners is not in $A$.

Keywords: Kronecker’s theorem, rational independence, track runners, multi-agent patrolling, idle time.

1 Introduction

In the classic lonely runners conjecture, introduced by Wills [11] and Cusick [4], $k$ agents run clockwise along a circle of length 1, starting from the same point at time $t = 0$. They have distinct but constant speeds. A runner is called lonely when he/she is at distance of at least $\frac{1}{k}$ from any other runner (along the circle). The conjecture asserts that each runner $a_i$ is lonely at some time $t_i \in (0, \infty)$. The conjecture has only been confirmed for up to $k = 7$ runners [1, 2]. A recent survey [7] lists a few other related problems.

Recently, some problems with similar flavor have appeared in the context of multi-agent patrolling, particularly in some one-dimensional scenarios [3, 5, 6, 9, 10]. Suppose that $k$ mobile agents with (possibly distinct) maximum speeds $v_i$ ($i = 1, \ldots, k$) are in charge of patrolling a closed or open fence (modeled by a circle or a line segment). The movement of the agents over the time interval $[0, \infty)$ is described by a patrolling schedule (or guarding schedule), where the speed of the $i$th agent, ($i = 1, \ldots, k$), may vary between zero and its maximum value $v_i$ in any of the two directions along the fence. Given a closed or open fence of length $\ell$ and maximum speeds $v_1, \ldots, v_k > 0$ of $k$ agents, the goal is to find a patrolling schedule that minimizes the idle time, defined as the longest time interval in $[0, \infty)$ during which some point along the fence remains unvisited, taken over all points. Several basic problems are open, such as the following: It is not known how to decide, given $v_1, \ldots, v_k > 0$, and $\ell, \tau > 0$ whether $k$ agents with these maximum speeds can ensure an idle time at most $\tau$ when patrolling a segment of length $\ell$.

This note is devoted to a question on track runners. In the spirit of the lonely runner conjecture, we posed the following question in [7]:

Assume that $k$ runners $1, 2, \ldots, k$, with distinct but constant speeds, run clockwise along a circle of length 1, starting from arbitrary points. Assume also that a certain half of the circular track (or any other fixed circular arc) is in the shade at all times. Does there exist a time when all runners are in the shade along the track?
Here we answer the question in the negative: the statement does not hold even if the shaded arc almost covers the entire track, e.g., has length 0.999, provided \( k \) is large enough.

**Notation and terminology.** We parameterize a circle of length \( \ell \) by the interval \([0, \ell]\), where the endpoints of the interval \([0, \ell]\) are identified. A *unit circle* is a circle of unit length \( C = [0, 1] \mod 1 \). A *schedule* of \( k \) agents consists of \( k \) functions \( f_i : [0, \infty) \to [0, \ell] \), for \( i = 1, \ldots, k \), where \( f_i(t) \) mod \( \ell \) is the position of agent \( i \) at time \( t \). Each function \( f_i \) is continuous, piecewise differentiable, and its derivative (speed) is bounded by \( |f_i'(t)| \leq v_i \). A schedule is called *periodic* with period \( T > 0 \) if \( f_i(t) = f_i(t + T) \mod \ell \) for all \( i = 1, \ldots, k \) and \( t \geq 0 \). \( H_n = \sum_{i=1}^{n} 1/i \) denotes the \( n \)th harmonic number; and \( H_0 = 0 \).

## 2 Track runners in the shade

We first show that the general answer to the problem posed in \([7]\) is negative:

**Theorem 1.** Consider the unit circle \( C \) and a circular arc \( A \subset C \) of length \( \ell = |A| < 1 \). Then there exists \( k = k(\ell) \in \mathbb{N} \), and a schedule for \( k \) runners with \( k \) distinct constant speeds, so that at any time \( t \geq 0 \), at least one of the \( k \) runners is in the complement \( C \setminus A \).

\[ \text{Proof.} \] Set \( v_i = i \) as the speed of agent \( i \), for \( i = 1, \ldots, k \), where \( k = k(\ell) \in \mathbb{N} \) is to be specified later. Assume, as we may, that \( C \setminus A = [0, a] \), for some \( a \in (0, 1) \). Let \( t_0 = 0 \). Since the speed of each agent is an integer multiple of the circle length \( \operatorname{len}(C) = 1 \), the resulting schedule is periodic and the period is 1. To ensure that at any \( t \geq 0 \), at least one agent is in \([0, a]\), it suffices to ensure this covering condition on the time interval \([0, 1]\), i.e., one period of the schedule. All agents start at time \( t = 0 \); however, it is convenient to specify their schedule with their positions at later time.

Agent 1 starts at point 0 at time 0; at time \( a \), its position is at \( a \) (exiting \([0, a]\) ). Agent 2 starts at point 0 at time \( a \); at time \( a + a/2 \), its position is at \( a \) (exiting \([0, a]\) ). Agent 3 starts at point 0 at time \( a + a/2 \); at time \( a + a/2 + a/3 \), its position is at \( a \) (exiting \([0, a]\) ). Subsequent agents are scheduled according to this pattern. For \( i = 1, \ldots, k \), agent \( i \) starts at point 0 at time \( aH_{i-1} \); at time \( aH_i \), its position is at \( a \) (exiting \([0, a]\) ). The schedules are given by the functions \( f_i(t) = it - iaH_{i-1} \) for \( i = 1, \ldots, k \).

The construction ensures that

1. agent \( i \) is in \([0, a]\) during the time interval \([t_{i-1}, t_i]\), for \( i = 1, \ldots, k \).
2. \( \bigcup_{i=1}^{k} [t_{i-1}, t_i] \supseteq [0, 1] \).

Indeed, condition 2 is \( aH_k \geq 1 \), or equivalently \( H_k \geq 1/a \). Since \( \ln k \leq H_k \), it suffices to have \( \ln k \geq 1/a \), or \( k \geq \exp(1/a) \), and the theorem is proved. \( \square \)

Now that we have seen that the general answer is negative, it is however interesting to exhibit some scenarios when the result holds.

A set of numbers \( \xi_1, \xi_2, \ldots, \xi_k \) are said to be *rationally independent* if no linear relation

\[ a_1\xi_1 + a_2\xi_2 + \cdots + a_k\xi_k = 0, \]

with integer coefficients, not all of which are zero, holds. In particular, if \( \xi_1, \xi_2, \ldots, \xi_k \) are rationally independent, then they are pairwise distinct. Recall now Kronecker’s theorem; see, e.g., \([8, \text{Theorem} 444, \text{p.} 382]\).
Theorem 2. (Kronecker, 1884) If $\xi_1, \xi_2, \ldots, \xi_k \in \mathbb{R}$ are rationally independent, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ are arbitrary, and $T$ and $\varepsilon$ are positive reals, then there is a real number $t > T$, and integers $p_1, p_2, \ldots, p_k$, such that
\[ |t\xi_m - p_m - \alpha_m| \leq \varepsilon \quad (m = 1, 2, \ldots, k). \]

As a corollary, we obtain the following result.

Theorem 3. Assume that $k$ runners $1, 2, \ldots, k$, with constant rationally independent (thus distinct) speeds $\xi_1, \xi_2, \ldots, \xi_k$, run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc $A \subset C$ and for every $T > 0$, there exists $t > T$ such that all runners are in $A$ at time $t$.

Proof. Assume, as we may, that $A = [0, a]$, for some $a \in (0, 1)$. Let $0 \leq \beta_i < 1$, be the start position of runner $i$, for $i = 1, 2, \ldots, k$. Set $\alpha_i = a/2 + 1 - \beta_i$, for $i = 1, 2, \ldots, k$, set $\varepsilon = a/3$, and employ Theorem 2 to finish the proof. \qed

Remark. It is interesting to note that Theorem 1 gives a negative answer regardless of how long the shaded arc is, while Theorem 3 gives a positive answer regardless of how short the shaded arc is and for how far in the future one desires.

Observe that if $\xi_1, \xi_2, \ldots, \xi_k$ are rationally independent reals, then at least one $\xi_i$ must be irrational (in fact, all but at most one $\xi_i$ must be irrational). To obtain the conclusion of Theorem 3 neither the condition that the speeds $\xi_1, \xi_2, \ldots, \xi_k$ are rationally independent, nor the condition that at least one $\xi_i$ is irrational are necessary. For instance, a condition imposed on the relative speeds suffices as it is shown in the following.

Theorem 4. Assume that $k$ runners $1, 2, \ldots, k$, with constant but distinct speeds run clockwise along a circle of length 1, starting from arbitrary points. For every circular arc $A \subset C$, there exist suitable distinct speeds $v_1, v_2, \ldots, v_k > 0$, so that for every $T > 0$, there exists $t > T$ such that all runners are in $A$ at time $t$.

Proof. Assume, as we may, that $A = [0, a]$, for some $a \in (0, 1)$. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the starting points of the runners, where $0 \leq \beta_i < 1$, for $i = 1, 2, \ldots, k$. We proceed by induction on the number of runners $k$, and with a stronger induction hypothesis extending to every arc $A$. The base case $k = 1$ is satisfied by setting $v_1 = 1$ for any interval. The subsequent speeds will be set to increasing values, so that $v_1 < v_2 < \cdots < v_k$.

For the induction step, assume that the statement holds for runners $1, 2, \ldots, k - 1$, the arc $A' = [0, a/2]$ and $T$, and we need to prove it for runners $1, 2, \ldots, k$, the arc $A = [0, a]$ and $T$. By the induction hypothesis, there exists $t > T$ so that runners $1, 2, \ldots, k - 1$, are in $A'$ at time $t$. Set $v_k = \frac{2}{a}v_{k-1}$. Observe that runner $k$ will enter the arc $A$ at point $0$ before any of the first $k - 1$ runners exits $A$ at point $a$, regardless of his or her starting point. Hence all $k$ runners will be in $A$ at some time in the interval $[t, t + 1/v_k]$, completing the induction step, and thereby the proof of the theorem. \qed

References


