Offline Variants of the "Lion and Man" Problem [1]

— Some problems and techniques for measuring crowdedness and for safe path planning —

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Abstract

Consider the following *safe path planning* problem: Given a set of trajectories (paths) of k point robots with maximum unit speed in a bounded region over a (long) time interval $[0, T]$, find another trajectory (if it exists) subject to the same maximum unit speed limit, that *avoids* (that is, stays at a safe distance of) each of the other k trajectories over the entire time interval. We call this variant the *continuous model* of the safe path planning problem. The *discrete model* of this problem is: Given a set of trajectories (paths) of k point robots in a graph over a (long) time interval $0, 1, 2, \ldots, T$, find a trajectory (path) for another robot, that avoids each of the other k at any time instance in the given time interval.

We introduce the notions of the *avoidance number* of a region, and that of a graph, respectively, as the maximum number of trajectories which can be avoided in the region (respectively, graph). We give the first estimates on the avoidance number of the $n \times n$ grid G_n, and also devise an efficient algorithm for the corresponding *safe path planning* problem in arbitrary graphs. We then show that our estimates on the avoidance number of G_n can be extended for the avoidance number of a bounded (fat) region. In the final part of our paper, we consider other related offline questions, such as the *maximum number of men* problem and the *spy* problem.

Key words: Lion and man problem, safe path planning, avoidance number, measuring crowdedness.

1 Introduction

Suppose you are in a hall of a train station for a long period of time and plan to stay safe. There are k other people moving around in the train station and you want to stay at least one meter away from each person. Your maximum speed and any other moving person's maximum speed is 1 meter per second. Even for $k = 1$, the solution to David Gale's lion and man problem [2,7,12] in the continuous model shows that one cannot hope for a survival solution in this *online* game. If however, the train station hall has a pillar in the center, then it is easy to avoid one person (but not two) by moving around the pillar. Suppose therefore that you are given the trajectories of each of the k other persons over the long time interval; can you now find a safe trajectory (a path with start and end position at your choice) over the same time interval, so that the distance from you to any other person is at least one at each time instance? Instead of the train station, suppose now you have the same goal while wandering on the streets of Manhattan. Again, even with only two other people moving in the city ($k = 2$), it is not hard to see that one cannot hope to find a safe trajectory (path) in this online game over a long enough period of time. Can one find such a trajectory when one already knows the complete trajectories over the entire time interval of the other people moving around?

Let us briefly recall the "lion and man" problem. The (time) continuous version, attributed to Rado, goes as follows: A man M and a lion L are moving within a given area (a closed arena), both having equal maximum (unit) speeds. The lion wins if he catches the man. The man wins if he can keep escaping forever. The question is whether there exist tactics which guarantee capturing the man by the lion in finite time. Somewhat surprisingly, the answer has been shown to be negative by Besicovitch, who proved that an infinite polygonal path can be constructed for M to run along so that he is not captured in finite time. Besicovitch's solution was presented by Littlewood in his book [12] (pp. 135); see also [5] (pp. 45–47). Croft [7] and Sgall [17] considered other variations of the problem.

One can observe that in Besicovitch's solution, although the man can escape indefinitely, the distance between the man and the lion becomes arbitrarily small over time. Therefore a natural follow-up question was: whether the man could also maintain a "safety radius" around him while trying to stay away from the lion. Alonso et al. [2] showed that this is not possible: if a lion

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pursues a man inside a circular region of radius r, then the lion can get within a distance c of the man in time $O(r \log \frac{r}{c})$. Their solution also extends to the case of pursuit in a square region, for instance. Inspired by these results, we study this scenario in the offline setting, that is, we wish to avoid (keep a safe distance from) a set of known (planned) trajectories over an arbitrary long time interval.

We first introduce the avoidance question for a given region in the *continuous model*. Let R be a connected bounded region in the plane (for instance a rectangle). For $T \geq 0$, let $\pi : [0, T] \to R$ be the continuous function describing a *trajectory* or a *path*, that is, $\pi(t)$ denotes the position of a player (the lion or man) along the trajectory at time t. Any path considered is subject to the maximum unit speed restriction. Given two paths ρ, μ over the interval $[0, T]$, we say that ρ and μ *avoid* each other if at any time $t \in [0, T]$, the Euclidean distance $d(\rho(t), \mu(t))$ is at least 1. If the two paths ρ and μ do not avoid each other, we say that they *intersect*, or *collide*. We are interested in questions like:

Safe path planning problem in a region. Given a set of k paths (trajectories) over the time interval $[0, T]$ in a region R, does there exist another path in R over the same time interval which avoids the k given paths over the time interval $[0, T]$? Compute such a path if it exists. Variant: Find such a path when two distinguished start and target points, or one of them, are specified.

Avoidance number of a region. The *avoidance number* $\kappa(R)$ of a region R is the maximum number of *arbitrary* paths (trajectories) in R that can be avoided over an arbitrary long time interval $[0, T]$, that is, every set of $\kappa(R)$ paths can be avoided, and there exists a set of $\kappa(R) + 1$ paths that cannot be avoided.

Note that $\kappa(R)$ is well-defined, since placing a sufficient number of stationary lions in R makes it impossible to add another path that avoids all of them. For instance, let R be a rectangle with dimensions $N \leq M$. Consider the example of $\lfloor N/2 \rfloor + 1$ unavoidable parallel paths which sweep R: they start simultaneously at one of the sides of length N and traverse at (maximum) unit speed all the way to the opposite side of length N; this yields that $\kappa(R) \leq \lfloor N/2 \rfloor$ for a rectangle R with dimensions $N \leq M$. Also, it can be assumed that R is connected, since knowing $\kappa(R')$ for each connected part easily allows to obtain $\kappa(R)$.

In many cases, geometric problems are reduced to their discrete counterparts in order to simplify the considerations and then to extend the results to arbitrary regions. Following this, we now introduce the avoidance question in graphs in the *discrete model*.

Let $G = (V, E)$ be a given graph. For a vertex $v \in V$, the *open neighborhood* of

v, denoted $N(v)$, is the set of all neighbors of v, and the *closed neighborhood* of v is defined as $N[v] := N(v) \cup \{v\}$. For a given positive integer T, let $[0, T]$ denote the (discrete) time interval $[0, T] = \{0, 1, \ldots, T\}$. A *path* (or *trajectory*) in G over the time interval $[0, T]$ is a sequence of vertices $\pi = \pi_0, \pi_1, \ldots, \pi_T$, where $\pi_{i+1} \in N[\pi_i]$ for each $i = 0, \ldots, T-1$; i.e., consecutive vertices in the path are either the same or adjacent vertices in the graph. Given two paths ρ, μ over the interval $[0, T]$, we say that ρ and μ *avoid* each other if at any time $t \in \{0, 1, \ldots, T\}$ the two paths are at different vertices, and if the two paths never "traverse" the same edge from opposite directions when moving from their positions at time t to their positions at time $t+1$. If the two paths ρ and μ do not avoid each other, we say that they *intersect*, or *collide*. We are interested in the following questions:

Safe path planning problem in a graph. Given a set of k paths (trajectories) over the time interval $[0, T]$ in a graph $G = (V, E)$, does there exist another path in G over the same time interval which avoids the k given paths over the time interval $[0, T]$? Compute such a path if it exists. Variant: Find such a path when two distinguished start and target vertices, or only one of them, are specified.

Avoidance number of a graph. The *avoidance number* $k(G)$ of a graph G is the maximum number of *arbitrary* paths in R that can be avoided over an arbitrary long time interval $[0, T]$. (Every set of $k(G)$ paths can be avoided, and there exist $k(G)+1$ paths which cannot be avoided.) Given a graph G, how hard is it to compute $k(G)$?

Similarly as for regions, one can assume G is connected, since knowing $k(G')$ for each connected component G' of G easily allows to obtain $k(G)$. For instance $k(C_n) = 1$, and $k(P_n) = 0$, where C_n and P_n are the cycle, respectively path, on n vertices.

Our discrete problem differs form the classical cop-and-robber games, introduced by Nowakowski and Winkler [14], in that all players have "full vision" and thus know the positions of the others at all times. The problem also differs form the classical graph search problems surveyed by, e.g. Bienstock [4] or Fomin and Thilicos [11], where a searcher may jump to an arbitrary vertex, and the fugitive is allowed to move at infinite speed; a variant where the fugitive moves to positions of bounded distance is studied by Dendris et al. [9], however, the fugitive is additionally restricted to move only when a searcher occupies an adjacent vertex. To the best of our knowledge, only Fomin et al. [10] and Petrov [15] study a model similar to ours for the graph formed by the edges of a tree or those of a regular polyhedron. In a forthcoming paper [8], we study the problem of pursuit-evasion in the grid G_n in the line-of-sight vision model, in the online setting.

1.1 Our results

In Section 2, we give first estimates on the avoidance number of the $n \times n$ grid G_n, namely, we prove $k(G_n) = \Omega(\sqrt{n})$. In Section 3, we present efficient algorithms for the safe path planning problem in G_n, and we then extend this result for arbitrary graphs. In Section 4, we show that our estimates on the avoidance number of G_n can be extended for the avoidance number of a bounded (square) region. Finally, we consider other related offline questions, such as the *maximum number of men* problem and the *spy* problem (Section 5).

1.2 Applications

These questions help us in defining and measuring *crowdedness*. Take for instance the continuous model with maximum unit speed limitation in a square or the discrete model in a grid. A set Π of maximum unit speed trajectories (paths) over a time interval $[0, T]$ in a bounded region R is said to be *crowded* if no other path ρ exists such that ρ avoids each member of Π over $[0, T]$. Let us point some applications. Measuring and estimating air or sea traffic congestion (or that on a system of roads) and understanding how it builds up have become quite challenging problems in the modern and most likely tomorrow's world. Not in a far away future, robot cars and other systems of autonomous robots moving around could become a casual encounter. Assume that we have a system of robots moving in a confined area according to planned trajectories. The system that controls them must be able to answer questions like: can another robot be added without causing deadlock or congestion? If yes what should be its path? All of these problems deal in a way or another with measuring how crowded is the given region under the current schedule, and whether adding new members into the traffic will cause congestion or not.

Avoidance problems of a similar spirit have been studied for instance by Reif and Sharir [16], who investigate the computational complexity of planning the motion of a body B in 2-D or 3-D space, so as to avoid collision with moving obstacles of known trajectories. The 3-D dynamic movement problem has been shown to be intractable even if B has only a constant number of degrees of freedom of motion. In one of their variants called the *asteroid avoidance problem*, B is a convex polyhedron that can move by translation with bounded speed, while the obstacles have known translational trajectories. As pointed out in [16], this problem has many applications to robot, automobile, and aircraft collision avoidance. While in their variants the geometry of the moving objects plays an important role, for us the geometry of the region plays a similar role. More generally, the environment is abstracted as a graph, and the moving robots are abstracted to moving points. Another difference is our interest in the combinatorial aspect (in knowing the maximum number of trajectories that can be avoided) besides the computational one.

2 The avoidance number of a grid

The $n \times n$ *grid graph* (or just *grid*) $G_n = (V, E)$ is the set of n^2 vertices (points) with integer coordinates in $[0, n-1] \times [0, n-1]$ together with their connecting edges, where $\{(x_1, y_1), (x_2, y_2)\} \in E$ if either $x_1 = x_2$ and $|y_1 - y_2| = 1$ or $|x_1 - x_2| = 1$ and $y_1 = y_2$. In this section, we deal with the discrete version in the $n \times n$ grid G_n and estimate $k(n) = k(G_n)$, the maximum number of paths that can be always avoided over an arbitrary long time interval.

A few simple observations show that for any $n \geq 2$, we have: (i) $k(n) \geq 1$, and (ii) $k(n) \leq n - 1$. To verify the first, let π be the given path, and let ρ be the path we want; arbitrarily select one "block" in the grid, for example at the North-West corner (assuming without loss of generality that π does not start at any of the four grid points of that block). Look in advance at the moment (time) t when π reaches one of these grid points, say p. If this never happens, ρ can safely stay at one of these points. Otherwise, ρ will start at the grid point q opposite to p with respect to the center of the block and follow the moves of π so that it remains at a point opposite to that of π's current position. If π happens to leave the block, ρ can still ensure that when π comes back to the block again, it will be on the opposite grid point, etc.

To verify the second, take n horizontal paths starting simultaneously at the leftmost street (left border of G_n) and sweeping right until they all reach the rightmost street (right border of G_n) at time $T = n$. Clearly, no safe path which avoids the n given paths exists. Although showing that $k(n) \geq 1$ is immediate, to show that $k(n) \geq 2$ (for large enough n) requires some thought; the reader may want to try this a little bit before proceeding in order to familiarize himself with the problem.

Proposition 1 *If $n \geq 4$, two paths can be avoided in G_n, that is, $k(n) \geq 2$.*

PROOF. Let π^A and π^B be the given paths of moving points A and B over $[0, T]$ in G_n. The idea of the proof is to divide the time interval $[0, T]$ into smaller intervals and extend the avoiding path π^Z incrementally in each interval until we reach time T; the time intervals (increments) are of length 1 or 3. More precisely, we start our path π^Z of Z at a "free" grid position p at time $t = 0$ that does not lie on the boundary of G_n. Clearly, there exists such p as $n \geq 4$ and only two paths are to be avoided. We maintain the following invariant:

- the current (partial) path of Z over the interval $[0, t]$ avoids π^A and π^B on this interval, and
- π_t^Z is not on the boundary of G_n.

We then show that path π^Z can be extended either with one more step or with three more steps so that the invariant is maintained. Clearly, the invariant holds initially (at time $t = 0$) by the choice of the start point of Z, so assume that the invariant holds at time t. There are two cases to consider:

Case 1: neither A or B "attacks" the current position π_t^Z of Z at time $t+1$, that is, $\pi_{t+1}^A \neq \pi_t^Z$ and $\pi_{t+1}^B \neq \pi_t^Z$. Then we set $\pi_{t+1}^Z = \pi_t^Z$, and the invariant is maintained at time $t+1$ by the induction hypothesis and by the assumption on points π_{t+1}^A and π_{t+1}^B.

Case 2: at least one of A and B "attacks" the current position π_t^Z of Z at time $t+1$, that is, either $\pi_{t+1}^A = \pi_t^Z$ or $\pi_{t+1}^B = \pi_t^Z$. Then, if there exists a point p not on the boundary of grid G_n such that $\pi_{t+1}^A \neq p$, $\pi_{t+1}^B \neq p$, and $p \in N(\pi_t^Z)$, we may set $\pi_{t+1}^Z = p$, and the invariant is maintained at time $t+1$ by the definition of point p. Otherwise, the lack of such point p implies

$$\pi_t^Z \in \{(1,1), (1, n-1), (n-1, n-1), (n-1, 1)\},$$

and:

(a) either $\pi_{t+1}^A = \pi_{t+1}^B = \pi_t^Z$, $\pi_t^A \neq \pi_t^B$, and both points π_t^A and π_t^B are not on the boundary of G_n (see Fig. 1);
(b) or $\pi_{t+1}^A = \pi_t^Z$, $\pi_t^A \neq \pi_t^B$, π_t^A is not on the boundary of G_n, and $\pi_{t+1}^B \in N(\pi_t^Z)$;
(c) or $\pi_{t+1}^B = \pi_t^Z$, $\pi_t^A \neq \pi_t^B$, π_t^B is not on the boundary of G_n, and $\pi_{t+1}^A \in N(\pi_t^Z)$.

Subcases (b) and (c) are symmetric, thus if w.l.o.g. we assume that $\pi_t^Z = (1,1)$, then there are only two subcases to consider (modulo a relabeling of paths).

Subcase (a): $\pi_{t+1}^A = \pi_{t+1}^B = (1,1)$, $\pi_t^A = (1,2)$, and $\pi_t^B = (2,1)$. For simplicity, we shall describe each of paths π^A, π^B and π^Z during three consecutive steps starting from π_{t+1}^A, π_{t+1}^B and π_{t+1}^Z, respectively, as a sequence of length 3 over the alphabet Σ ={N,E,S,W,Y}, where the first four are the standard grid orientations, and the last one 'Y' is for "stay". Then Z moves according to the following subcases:

- $\pi_{t+3}^B = (1,2)$ or B moves WNW ($\pi_{t+1}^B = (1,1)$, $\pi_{t+2}^B = (1,2)$, and $\pi_{t+3}^B = (0,2)$). Then, if $\pi_{t+3}^A \in \{(2,0),(2,1)\}$, then Z moves WYE, otherwise, Z moves SEN.
- B moves WWN. Then, if $\pi_{t+3}^A \in \{(2,0),(2,1)\}$, then Z moves WNE, otherwise, Z moves SEN.
- A moves neither SYN, SNY, nor SNW, and $\pi_{t+3}^B \notin \{(0,2),(1,2)\}$. Then Z moves WNE.

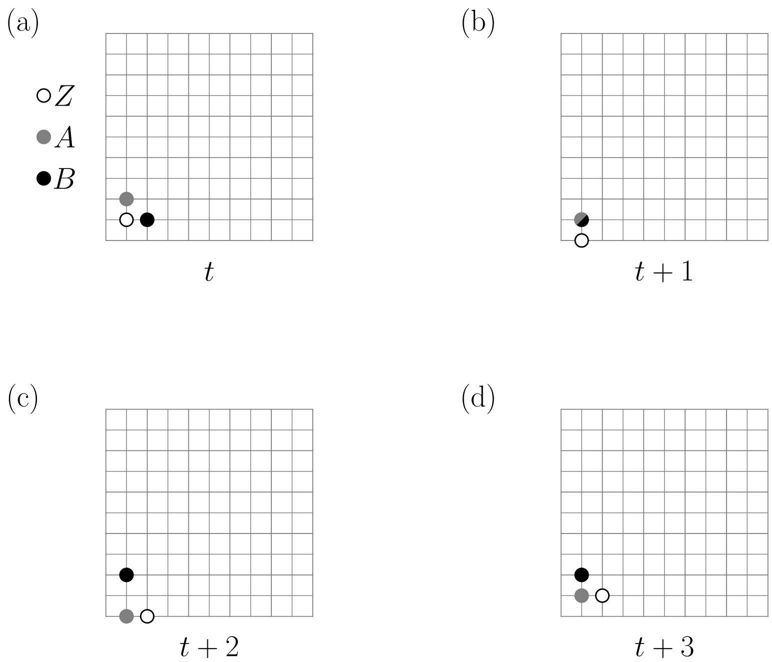


Fig. 1. Subcase (a): extending path π^Z on the current time interval by 3 steps when A moves SSN, B moves WNY ($\pi^A_{t+1} = \pi^B_{t+1} = (1,1)$): Z moves SEN.

- A moves SYN, SNY, or SNW, and $\pi^B_{t+3} \notin \{(0,2),(1,2)\}$. Then, if B moves neither WYE, WEY, nor WES, then Z moves SEN, otherwise, Z moves WYE.

The above analysis exhausts all possible cases, and the invariant is maintained at time $t+3$.

Subcase (b): $\pi^A_{t+1} = (1,1)$, $\pi^A_t = (1,2)$, and $\pi^B_{t+1} = (2,1)$. Similarly as above, Z can move according to the following subcases:

- $\pi^B_{t+3} = (1,2)$. Then, if $\pi^A_{t+3} \in \{(2,0),(2,1)\}$, then Z moves WYE, otherwise, Z moves SEN.
- A moves neither SYN, SNY, nor SNW, and $\pi^B_{t+3} \notin \{(0,2),(1,2)\}$. Then Z moves WNE.
- A moves SYN, SNY, or SNW, and $\pi^B_{t+3} \notin \{(0,2),(1,2)\}$. Then, if the second and third step of B is neither WE, NS, EW, SN, YY, YS, SY, SW, nor SE, then Z moves SEN, otherwise, Z moves WYE.

The above analysis is exhaustive, therefore the invariant is maintained at time $t+3$. □

The next result shows that $\Omega(\sqrt{n})$ paths can be avoided in G_n. Our path construction extends a partial path in an iterative manner; the extension in each iteration is obtained in a probabilistic way.

Theorem 2 *$\Omega(\sqrt{n})$ paths can be avoided in G_n. Thus we have $\Omega(\sqrt{n}) = k(n) \leq n - 1$.*

PROOF. Let $\pi^1, \ldots, \pi^k$ be the k given paths over $[0, T]$. The plan to construct a path π^Z which avoids them is a generalization of the argument used to prove $k(n) \geq 2$ (in Proposition 1). There we divided the time interval $[0, T]$ into smaller intervals and extended the avoiding path π^Z incrementally in each interval until we reached time T; the time intervals (increments) were of length 1 or 3. Here we use (longer) intervals of length $2n$.

Without loss of generality we can assume that T is a multiple of $2n$. (Otherwise extend the k given paths arbitrarily, say for each, by staying at the final position, until the next time step which is a multiple of $2n$. Then plan to avoid these extended k paths on the extended time interval.) We divide the time interval $[0, T]$ into smaller intervals of length $2n$, called *rounds*, and divide correspondingly each of paths $\pi^1, \ldots, \pi^k$ into $(2n+1)$-vertex subpaths, with the final point of a subpath being the same as the initial point of the next subpath. Here the value $2n$ is chosen so that it permits reaching any point from any other point in the grid G_n within this number of steps. We extend the path π^Z incrementally in each such round (time interval of length $2n$) until we reach time T.

For $r > 0$, denote by $B(p, r)$ the r-ball centered at a grid point p in the L_1 norm. Let $A(r)$ be the maximum number of grid points in $B(p, r)$. We have $A(r) \leq 2(r+1)^2 - 2(r+1) + 1$; note that $r^2 \leq A(r) \leq 3r^2$ for any $r \geq 3$. Put $q = 1/A(r)$. We use the following two parameters k and r, where k is the number of given paths and r is the radius of a ball in the L_1 norm. The two parameters are chosen to satisfy $kr^2 \approx n^2$ and $nk \approx r^2$; more precisely,

$$k = \lfloor c_1 n^{1/2} \rfloor, \quad r = c_2 n^{3/4}, \tag{1}$$

where the constants c_1 and c_2 will be specified later in order to satisfy certain inequality constraints (see the end of this section).

We now show how to select the start point s of the subpath $\widetilde{\pi}^Z$ of π^Z in the current round. For the first round, select s such that (i) $B(s, r)$ is contained in G_n and (ii) $B(s, 3r)$ does not contain any of the k start points of $\pi^1, \ldots, \pi^k$. For any subsequent round we proceed as follows. Let g be the chosen target position from the previous round; g is such that

- $B(g, r)$ is contained in G_n and
- $B(g, 3r)$ does not contain any of the k final points of the subpaths of $\pi^1, \ldots, \pi^k$ in the previous round.

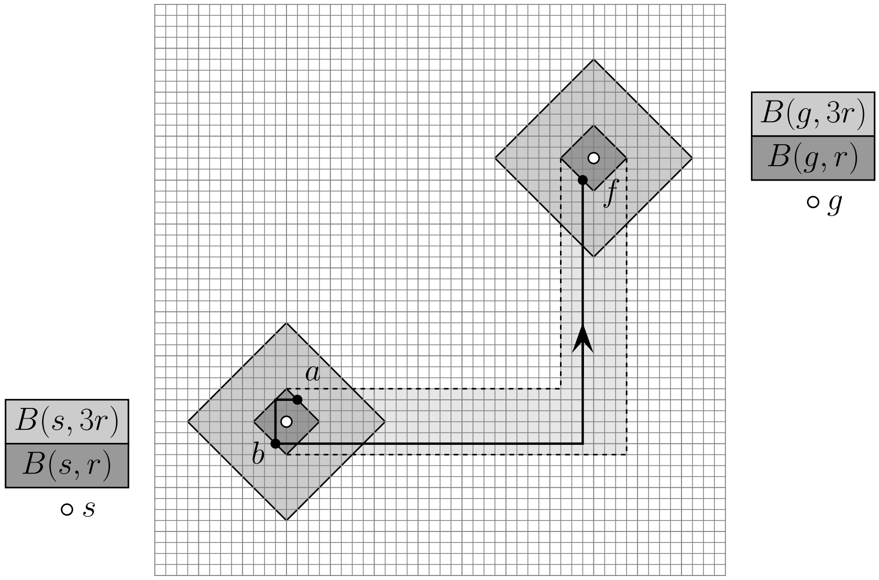


Fig. 2. Extending the path π^Z in the current time interval: $\widetilde{\pi}^Z = a, \ldots, b, \ldots, f$; here $r = 3$.

We refer to these two conditions as the *emptiness property* of $B(g, 3r)$. Let f be the final point of the subpath of Z in the previous round. We will ensure that f lies in $B(g, r)$. Set now $a := f$ and $s := g$. In the current round, we extend our path π^Z of Z from start position a in $B(s, r)$; for the first round, $a = s$. See Figure 2.

We will maintain the following invariant after each round: the current (partial) path of Z over the interval $[0, 2in]$ avoids $\pi^1, \ldots, \pi^k$ on this interval, and the final position of this path lies in $B(g, r)$ for some point g that satisfies the emptiness condition. We will then show that path π^Z can be extended with $2n$ steps so that the invariant is maintained. Note that the invariant holds initially (at time $t = 0$) by the choice of the start point of Z.

Assume the invariant holds at time $t = 2in$. Set now $a := f$ and $s := g$. In the current round, we extend our path π^Z of Z from start position a in $B(s, r)$. First, select a new target position g for the subpath starting at a on $[2in, 2(i+1)n]$. The target g is chosen so that $B(g, 3r)$ satisfies the emptiness property. The argument justifying the possibility of such a choice in each round is delayed till the end of the proof, namely the derivation leading to inequalities (4) and (5). Let (x_g, y_g) be its coordinates. For simplicity and w.l.o.g., at the beginning of each round of $2n$ steps we imagine that:

(i) we rotate the coordinate axes such that after rotation the target position dominates the start position, i.e., $x_g \geq x_s$ and $y_g \geq y_s$, and
(ii) we reset the clock so that we have the time running in the interval $[0, 2n]$

instead of $[2in, 2(i+1)n]$.

Select uniformly at random an intermediate target b in $B(s,r)$. We extend π^Z in two phases:

PHASE 1: In at most $2r$ steps, Z moves from a to the selected intermediate target b and then stays there until time $t_0 = 2r$.

PHASE 2: Z continues from b with an L-walk for the next $(2n - t_0)$ steps towards $f = g + (b - s) \in B(g,r)$: Z moves at maximum unit speed right for the first $(x_g - x_s)$ steps, then up for the next $(y_g - y_s)$ steps. Z stays at f until $t = 2n$ for the remaining $2n - t_0 - (x_g - x_s) - (y_g - y_s)$ steps.

In the first $2r$ steps of the round, Z moves from a to the randomly chosen intermediate target b in $B(s,r)$; then waits there until time $t_0 = 2r$. By the emptiness property of $B(s,3r)$ — note, this is $B(g,3r)$ from the previous round — all k initial subpaths of $\pi^1, \ldots, \pi^k$ in the first $2r$ steps remain outside $B(s,r)$, so neither intersects the initial subpath of Z in this time frame of $2r$ steps. Denote by $\mathrm{Prob}(Z_t = z)$ the probability that the position of Z at time step t is the grid point z. Recall that $q = 1/A(r)$. Note that at time t_0:

$$\mathrm{Prob}(Z_{t_0} = z) = \begin{cases} q \text{ if } z \in B(s,r), \\ 0 \text{ if } z \in G_n \setminus B(s,r) \end{cases}$$

In particular, we have $\mathrm{Prob}(Z_{t_0} = z) \le q$ for any $z \in G_n$. Next, since the intermediate target b is selected uniformly at random with probability q, for any $t \in [t_0, 2n]$ and $z \in G_n$, among all possible L-walks starting at points of $B(s,r)$ at $t = t_0$, there exists at most one walk π such that $\pi_t = z$. Consequently, we obtain

$$\mathrm{Prob}(Z_t = z) \le q \text{ for } z \in G_n,\ t \in [t_0, 2n]. \tag{2}$$

We now bound from above the probability of the "bad" event that the subpath $\widetilde{\pi}^Z$ of π^Z intersects any of the other k given subpaths in this round. For a given $j \in \{1, \ldots, k\}$, let $L = L(j)$ be the moving point on the given subpath $\widetilde{\pi}^j$ (to simplify notation, we omit the index j), and let L_t be the position of L at time $t \in [0, 2n]$ (the current round) on the given subpath $\widetilde{\pi}^j$.

$$\begin{aligned} \mathrm{Prob}(\widetilde{\pi}^Z \text{ intersects } \widetilde{\pi}^j) &\le \sum_{t=t_0+1}^{2n} \mathrm{Prob}(Z_t = L_t) + \sum_{t=t_0+1}^{2n-1} \mathrm{Prob}(Z_t = L_{t+1}) \\ &\le 2nq + 2nq = 4nq. \end{aligned}$$

The first sum above bounds the probability that the two paths coincide at some grid point in G_n, while the second sum bounds the probability that the two paths intersect by crossing the same edge of G_n from opposite directions; inequality (2) is used in both. By summing up over the k paths, and letting $P = \text{Prob}(\widetilde{\pi}^Z \text{ intersects some } \widetilde{\pi}^j)$, the union bound gives

$$P \le 4nkq \le \frac{4nk}{r^2} \le \frac{4c_1}{c_2^2}, \tag{3}$$

where c_1 and c_2 are as introduced in (1).

We will later ensure that $P < 1$. We now argue that our choice of parameters k and r allows to maintain the emptiness property at each round. The total area of the k balls of radius $3r$ centered at the k final positions of a round ($t = 2n$) is at most $kA(3r) \le 27kr^2$. If the following inequality is satisfied

$$27kr^2 \le n^2 - 4 \cdot r \cdot n = n^2 - 4rn, \tag{4}$$

then there exists a "safe target" position g for the current round: that is, (i) $B(g, r)$ is contained in G_n and (ii) $B(g, 3r)$ does not contain any of the k final points of the k subpaths of $\pi^1, \ldots, \pi^k$. Inequality (4) amounts to having

$$n \ge \left(\frac{4c_2}{1 - 27c_1c_2^2} \right)^4.$$

Therefore satisfying

$$\begin{cases} \frac{4c_1}{c_2^2} < 1 \\ n \ge \left(\frac{3}{c_2}\right)^{4/3} \\ n \ge \left(\frac{4c_2}{1-27c_1c_2^2}\right)^4 \end{cases} \tag{5}$$

will ensure (i) $P < 1$, (ii) $r \ge 3$ and (iii) inequality (4). It is enough to take for instance (without making any attempt to optimize the constants) $c_1 = 1/39$, $c_2 = 1/3$, and $n \ge 20$.

Since $P \le 4c_1/c_2^2 = 36/39 < 1$, by the basic probabilistic method, the path of Z can be extended in the current round while maintaining the emptiness property invariant at the beginning of the next round; see e.g. [1,13] for an overview of the method and its applications. The procedure is repeated for $T/(2n)$ rounds until the entire path π^Z is obtained. This completes the proof of Theorem 2. □

Remarks. It seems that neither a random walk in the four grid directions starting at s, nor a directed monotone random walk towards a safe target region starting at s, will give the above result. This is the reason for choosing the two-phase randomized path construction.

Without any strong evidence we are still tempted to conjecture that $\Omega(n)$ paths can be avoided in the $n \times n$ grid G_n over any arbitrary long time interval; that is, $k(n) = \Theta(n)$.[4] So far we could not even rule out the stronger possibility that $k(n) = n - 1$.

3 Algorithmic results

In this section, we present efficient algorithms for safe path planning in grids, which we then extend to arbitrary graphs. First we consider the *decision* version of the safe path planning problem with specified endpoints, that is, given two endpoints s and g and k paths $\pi^1, \ldots, \pi^k$ over the time interval $[0, T]$ in the grid G_n, we only ask whether there exists a path ρ, such that $s = \rho_0$ and $g = \rho_T$, which avoids the k given paths over $[0, T]$. Variant: one or both of the start and end positions is unspecified. We then consider computing such a path if it exists. As a general note, we do not impose any upper bound on k in terms of n, in particular k could be much larger than n^2.

Theorem 3 *Given k paths of length T in G_n to be avoided*:

(1) *With or without specified endpoint(s), the decision version of the safe path planning problem in G_n can be solved in $O((n^2 + k) \cdot T)$ time and $O(n^2)$ space.*
(2) *With or without specified endpoint(s), the safe path planning problem in G_n can be solved in $O((n^2 + k) \cdot T \cdot \log T)$ time and $O(n^2)$ space, or in $O((n^2 + k) \cdot T)$ time and $O(n^2 \cdot T)$ space.*

PROOF. Let V be the set of n^2 vertices of G_n. We construct a directed graph $G_{n,T}$ by creating $T+1$ copies of V and arranging them in $T+1$ layers, where each layer V_t, $t \in \{0, 1, \ldots, T\}$, serves to indicate the locations of the k lions and the possible locations of the man at time t in G_n. We denote by v_t the copy of vertex $v \in V$ in layer V_t. For every pair of vertices $u, v \in V$ that are adjacent in G_n, we create two directed edges (u_t, v_{t+1}) and (v_t, u_{t+1}) between every pair of adjacent layers V_t and V_{t+1}, $t \in \{0, 1, \ldots, T-1\}$, so that a move of a lion (or the man) in G_n from u at time t to v at time $t+1$ corresponds to a move from u_t to v_{t+1} in $G_{n,T}$. In addition, we create a directed edge (u_t, u_{t+1}) for every $u \in V$ and $t \in \{0, 1, \ldots, T-1\}$, since a lion (or the man)

[4] After submission of our paper we learned that recently Berger et al. [3], and independently Braß et al. [6], announced having proved this conjecture: even $\lfloor n/2 \rfloor$ lion paths can be avoided.

is allowed to remain at its current vertex in G_n. From the above construction, it is immediate that a path in G_n in the time interval $[0, T]$ corresponds to a directed path from layer V_0 to layer V_T in $G_{n,T}$. Note that $G_{n,T}$ has $O(n^2 T)$ vertices and edges.

Now, we modify $G_{n,T}$ by deleting those vertices corresponding to the paths $\pi^1, \ldots, \pi^k$ of the k lions, together with the edges incident on them. Since the man is not allowed to cross, in the opposite direction, an edge that a lion is crossing, we also delete those edges (u_t, v_{t+1}) such that (v_t, u_{t+1}) belongs to one of the k lion paths in $G_{n,T}$. Let $G'_{n,T}$ be the resulting graph. The following observations are immediate.

(1) There exists a safe path of the man in G_n which avoids the lions if and only if there exists a directed path from V_0 to V_T in $G'_{n,T}$.
(2) There exists a safe path of the man in G_n which avoids the lions between given start and goal vertices s and g, if and only if there exists a directed path from $s_0 \in V_0$ to $g_T \in V_T$ in $G'_{n,T}$.

To test either of the above conditions on $G'_{n,T}$, one only needs to maintain two copies of V and identify, successively for time instants $t = 0, 1, \ldots, T$, those vertices that are reachable from some vertex in V_0 (or from s_0, if s is given). This can be done in $O((n^2 + k)T)$ time using $O(n^2)$ space. Obviously, a safe path can be generated in $O((n^2 + k)T)$ time using $O(n^2 T)$ space if we maintain the entire $G'_{n,T}$. It is however possible to reduce the space requirement to $O(n^2)$ at the expense of time by using divide-and-conquer: We first identify the set W of vertices in $V_{\lfloor \frac{T}{2} \rfloor}$ that are reachable from V_0 (or from s_0, if s is given). This can be done in $O((n^2 + k)T)$ time using $O(n^2)$ space. Also, within the same time and space bounds, we identify the set W' of vertices in $V_{\lfloor \frac{T}{2} \rfloor}$ that are reachable from V_T (or from g_T, if g is given) by following the directed edges backwards. We then arbitrarily choose a vertex $w \in W \cap W'$, and solve the two subproblems recursively, namely, that of generating a safe path from V_0 to w, and that of generating a safe path from w to V_T. (If $W \cap W' = \emptyset$ then no safe path exists from V_0 to V_T.) This yields a recurrence $f(T) \leq f(\lfloor T/2 \rfloor) + f(\lceil T/2 \rceil) + O((n^2 + k)T)$ for the running time f (in $f(u)$, u is the number of moving steps considered, which is initially T). This gives $f(T) = O((n^2 + k) \cdot T \cdot \log T)$. □

Our algorithm can be extended for solving the safe path planning problem in graphs:

Theorem 4 *Given graph $G = (V, E)$ and k paths of length T to be avoided:*

(1) *With or without specified endpoint(s), the decision version of the safe path planning problem in a graph can be solved in $O((|V| + |E| + k) \cdot T)$ time and*

$O(|V|)$ space.

(2) *With or without specified endpoint(s), the safe path planning problem in a graph can be solved in $O((|V| + |E| + k) \cdot T \cdot \log T)$ time and $O(|V|)$ space, or in $O((|V| + |E| + k) \cdot T)$ time and $O((|V| + |E|) \cdot T)$ space.*

PROOF. When considering the safe path planning problem in graph, all we need is to notice that the argument in the proof of the previous theorem carries over if we replace $G_{n,T}$ by a multi-layer graph consisting of $T+1$ copies of the vertices of G together with appropriate edges between the layers. Observe that the number of such connections between two consecutive layers is now $O(|E|)$. Consequently, the whole layer graph has $|V| \cdot T$ vertices and $O(|E| \cdot T)$ edges, which allows us to derive the claimed complexity bounds. □

4 Avoiding many lions in the square (in the continuous model)

We show that our result in Theorem 2 can be extended to the continuous model using the path planning technique from the discrete model (in grid G_n).

Theorem 5 *$\Omega(\sqrt{N})$ paths can be avoided in a square Q of side N over an arbitrary long period of time; that is, $\Omega(\sqrt{N}) = \kappa(Q) \leq \lfloor N/2 \rfloor$, for $N \geq N_0$ (N_0 an absolute constant). Both the set of k given paths, and the constructed path are subject to the same maximum unit speed limit.*

PROOF. For simplicity, assume that the side length $N = n - 1$ is an integer. Let $\pi^1, \ldots, \pi^k$ be the k given paths in an $(n-1) \times (n-1)$ square Q over time interval $[0, T]$. Consider the $n \times n$ grid G_n superimposed over Q. Divide the time interval $[0, T]$ into smaller intervals of length $2n$ called *rounds* and divide correspondingly $\pi^1, \ldots, \pi^k$ into subpaths, where the final point of a subpath is the same as the initial point of the next subpath. We extend path π^Z incrementally until we reach time T (w.l.o.g. T is a multiple of $2n$). π^Z has the same structure as in the discrete case, with the understanding that between times t and $t+1$ for integer t, Z moves continuously at maximum speed along the corresponding grid edge, and that π^Z is at a grid point at any $t \in \{0, 1, \ldots, T\}$.

Phase 1 and Phase 2 are the same as in the proof of Theorem 2. The only difference is that the emptiness property requires now a larger ball $B(g, 4r)$, instead of $B(g, 3r)$. (The reader may consult Fig. 2, interpreting the entire area as an $(n-1) \times (n-1)$ square with grid lines superimposed, and replacing the ball $B(g, 3r)$ by a slightly larger $B(g, 4r)$.) The reason is the k given continuous paths can move in an arbitrary direction (not only along grid edges) in the square, subject only to the maximum unit speed constraint, as opposed to π^Z which is restricted to grid edges. Inequality (2) still holds. The change is in the bound on the probability P of intersecting any of the k given paths. Consider a path π on time interval $[a,b]$, and denote by $V_{[a,b]}(\pi)$ the set of grid points in G_n which are within the (Euclidean) distance at most 2 from some point on path π; namely,

$$V_{[a,b]}(\pi) = \{z \in G_n : \text{ there exists } t \in [a,b] \text{ such that } d(z,\pi(t)) \leq 2\}.$$

By the maximum unit speed limit constraint, if $b-a=1$ then $|V_{[a,b]}(\pi)| \leq 18$, and moreover, if none of the two endpoints of a given grid edge $e = \{z_1, z_2\}$ is in $V_{[a,b]}(\pi)$, then for any $t \in [a,b]$, point π_t is at the distance at least 1 from edge e. Consequently, the subpath of π^Z along edge e (regardless of the traversing direction) on time interval [a,b] avoids π. By construction, the path π^Z on time interval $[t_0, 2n]$ (the current round) is an L-walk, i.e., it consists of at most $(2n - t_0)$ grid edges (some edges degenerate to single grid points). Thus

$$\pi^Z_{[t_0,2n]} = \bigcup_{t \in \{t_0, t_0+1, \ldots, 2n-1\}} e_t,$$

where e_t is a grid edge. Consider now a path π^j (one of the given k paths). Taking into account the observation made in the previous paragraph, if for every $t = t_0, t_0+1, \ldots, 2n-1$ none of the two endpoints of edge $e_t = \{\pi^Z_t, \pi^Z_{t+1}\}$ is in $V_{[t,t+1]}(\pi^j)$, then path π^Z avoids path π^j on time interval $[t_0, 2n]$ (recall that π^Z avoids π^j on time interval $[0, t_0]$ by the emptiness property of ball $B(s, 4r)$). Consequently,

$$\begin{aligned}
&\text{Prob}(\pi^Z \text{ intersects } \pi^j \text{ on } [t_0, 2n]) \leq \\
&\quad \sum_{t \in \{t_0, t_0+1, \ldots, 2n-1\}} \text{Prob}(\pi^Z_t \in V_{[t,t+1]}(\pi^j)) + \\
&\quad \sum_{t \in \{t_0, t_0+1, \ldots, 2n-1\}} \text{Prob}(\pi^Z_{t+1} \in V_{[t,t+1]}(\pi^j)) \leq 2 \cdot 2n \cdot 18q = 72nq.
\end{aligned}$$

By summing up over k paths, the union bound gives

$$P = \text{Prob}(\pi^Z \text{ intersects some } \pi^j) \leq 72nkq \leq \frac{72nk}{r^2} \leq \frac{72c_1}{c_2^2}. \tag{6}$$

Therefore inequality (3) holds in this similar form. The total area of the k balls of radius $4r$ centered at the k final positions in a round is at most $kA(4r) \leq 48kr^2$. To maintain the emptiness property at each round, we require now $48kr^2 \leq n^2 - 4rn$. Thus all needed inequalities are satisfied by imposing

$$\begin{cases} \frac{72c_1}{c_2^2} < 1 \\ n \geq \left(\frac{3}{c_2}\right)^{4/3} \\ n \geq \left(\frac{4c_2}{1-48c_1c_2^2}\right)^4 \end{cases} \tag{7}$$

and the result follows as in the discrete case by an appropriate choice of the parameters, for instance, $c_1 = 1/300$, $c_2 = 1/2$, and $n \geq 20$. One can also take $N_0 = 20$. □

For simplicity, we have made our analysis for the square, but our result can be easily extended to any *fat* convex region R of diameter N (i.e., a region for which the ratio of the radii of smallest enclosing disk and largest enclosed disk is bounded from above by a constant), whence $\kappa(R) = \Omega(\sqrt{N})$, for $N \geq N_0$, for any such region R.

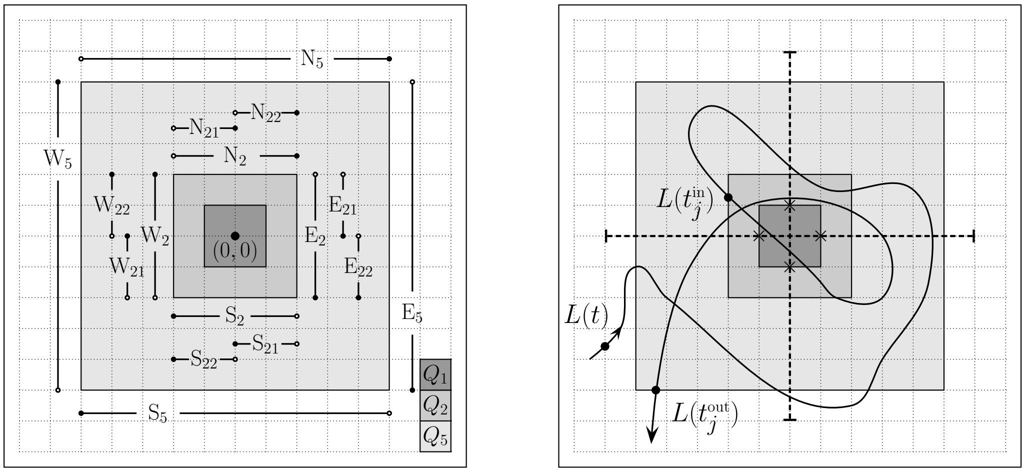
Corollary 6 *$\Omega(\sqrt{N})$ paths can be avoided in a fat region R of diameter N over an arbitrary long period of time; that is, $\Omega(\sqrt{N}) = \kappa(R) \leq \lfloor N/2 \rfloor$, for $N \geq N_0$ (N_0 an absolute constant). Both the set of k given paths, and the constructed path are subject to the same maximum unit speed limit.*

Similarly to the discrete model, we conjecture that in the continuous model, $\Omega(N)$ paths can be avoided in a square Q of size N over an arbitrary long time interval, that is, $\kappa(Q) = \Theta(N)$.

Theorem 5 gives that $\kappa(Q) \geq 1$ for $N \geq N_0$; for instance, with the above choice of the constants, $N \geq 300^2$ is needed to get $\kappa(Q) \geq 1$. Using a more direct argument, we obtain the following better constant in the bound:

Theorem 7 *For any $T \geq 0$, and any trajectory $L(t)$ of the lion in a square Q of side length $N \geq 12$, there exists a trajectory $M(t)$ of the man such that at any time $t \in [0, T]$, we have $d(M(t), L(t)) \geq 1$. That is, $\kappa(Q) \geq 1$ for $N \geq 12$.*

PROOF. Let $Q = [-N/2, N/2] \times [-N/2, N/2]$ be an $N \times N$ square, where $N \geq 12$. The vertices (corners) of S are $(-N/2, -N/2)$, $(N/2, -N/2)$, $(N/2, N/2)$ and $(-N/2, N/2)$. Let Q_i, $i = 1, 2, 5$, denote the $2i \times 2i$ square with corners $(-i, -i), (i, -i), (i, i)$ and $(-i, i)$, respectively, and let P denote the set of points $\{(0, 1)(-1, 0)(0, -1)(1, 0)\}$; these four points, called *initial points* will serve as initial positions for the man at various times. Next, for square Q_i, $i = 2, 5$, and $D \in \{\mathrm{N, W, S, E}\}$, define the D-side D_i of square Q_i as follows (for an illustration of terms defined in this paragraph, we refer to Fig. 3):



$Q = [-N/2, -N/2] \times [N/2, N/2]$

Fig. 3. An illustration of the definitions.

- N-side $\mathrm{N}_i = \{(x, i) \in Q_i : x \in (-i, i]\}$;
- W-side $\mathrm{W}_i = \{(-i, y) \in Q_i : y \in (-i, i]\}$;
- S-side $\mathrm{S}_i = \{(x, -i) \in Q_i : x \in [-i, i)\}$;
- E-side $\mathrm{E}_i = \{(i, y) \in Q_i : y \in [-i, i)\}$.

The union of points $\mathrm{N}_i \cup \mathrm{W}_i \cup \mathrm{S}_i \cup \mathrm{E}_i$ forms the boundary ∂Q_i of square Q_i, $i = 2, 5$, and the interior $\mathrm{Int}(Q_i)$ of square Q_i is $\mathrm{Int}(Q_i) = Q_i \setminus \partial Q_i$. Next, divide each of D-sides of square Q_2 into two parts:

- N-side $\mathrm{N}_2 = \mathrm{N}_{21} \cup \mathrm{N}_{22}$,
  where $\mathrm{N}_{21} = \{(x, 2) : x \in (-2, 0]\}$, and $\mathrm{N}_{22} = \{(x, 2) : x \in (0, 2]\}$;
- W-side $\mathrm{W}_2 = \mathrm{W}_{21} \cup \mathrm{W}_{22}$,
  where $\mathrm{W}_{21} = \{(-2, y) : y \in (-2, 0]\}$, and $\mathrm{W}_{22} = \{(-2, y) : y \in (0, 2]\}$;
- S-side $\mathrm{S}_2 = \mathrm{S}_{21} \cup \mathrm{S}_{22}$,
  where $\mathrm{S}_{21} = \{(x, -2) : x \in [0, 2)\}$, and $\mathrm{S}_{22} = \{(x, -2) : x \in [-2, 0)\}$;
- E-side $\mathrm{E}_2 = \mathrm{E}_{21} \cup \mathrm{E}_{22}$,
  where $\mathrm{E}_{21} = \{(2, y) : y \in [0, 2)\}$, and $\mathrm{E}_{22} = \{(x, 2) : y \in [-2, 0)\}$.

For $T \geq 0$, let $L : [0, T] \to Q$ be the function describing the trajectory of the lion, that is, $L(t)$ denotes the lion's position at time t; for simplicity, we can assume that both points $L(0)$ and $L(T)$ are points from the boundary ∂Q of the square Q — later this assumption will be removed. Analogously, we denote by $M : [0, T] \to Q$ the function describing the trajectory of the man, that is, $M(t)$ denotes man's position at time t. Recall that both the man and the lion are restricted to move with a maximum speed of 1, and the trajectory of M avoids that of L if for every $t \in [0, T]$, the Euclidean distance $d(L(t), M(t))$ between points $L(t)$ and $M(t)$ is at least 1.

Let $t_1^{\mathrm{in}}, \ldots, t_k^{\mathrm{in}} \in [0, T]$ be all last moments when the lion intersects the boundary ∂Q_2 of square Q_2 such that at the time moment $t_j^{\mathrm{out}} \in (t_j^{\mathrm{in}}, t_{j+1}^{\mathrm{in}})$ the lion intersects the boundary ∂Q_5 of square Q_5, and such that the lion intersects the boundary ∂Q_1 of square Q_1 between moments t_j^{in} and t_j^{out}. Formally:

- t_1^{in} is the last moment in $[0,T]$ such that $L(t_1^{\text{in}}) \cap \partial Q_2 \neq \emptyset$, and for each $t \in [0, t_1^{\text{in}}]$ we have $L(t_1^{\text{in}}) \cap \partial Q_1 = \emptyset$;
- if t_j^{out} is the smallest moment in $[t_j^{\text{in}}, T]$ such that $L(t_j^{\text{out}}) \cap \partial Q_5 \neq \emptyset$, then t_{j+1}^{in} is the last moment in $[t_j^{\text{out}}, T]$ such that $L(t_{j+1}^{\text{in}}) \cap \partial Q_2 \neq \emptyset$, and for each $t \in [t_j^{\text{out}}, t_{j+1}^{\text{in}}]$ we have $L(t_{j+1}^{\text{in}}) \cap \partial Q_1 = \emptyset$, $j = 1, \ldots, k$.

The idea is to define a strategy for the man according to time moments t_j^{in} and t_j^{out}. In general, if the lion is outside of square Q_5, then the man does nothing — just stays at point $(0,0)$ within square Q_1. If the lion gets to the area $Q_5 \setminus Q_1$, but does not reach the boundary ∂Q_1, the man still occupies point $(0,0)$ — clearly, for two points $p_1 \in Q_5 \setminus Q_1$ and $p_2 = (0,0)$ we have $d(p_1, p_2) > 1$. However, after the last time the lion crosses the boundary ∂Q_5 in order to get inside square Q_1, the man takes one of initial positions depending on D-sides of squares Q_2 and Q_5 which have points in common with $L(t_j^{\text{in}})$ and $L(t_j^{\text{out}})$, respectively. The properly chosen initial position will guarantee that the man will avoid the lion moving only either along the x- or y-axis within either the horizontal or vertical distance at least one from the lion, and moreover, before the lion intersects the boundary ∂Q_5, the man will be able to get inside square Q_2, and then to point $(0,0)$, which will allow us to proceed with the invariant proof.

Let us formalize the man's strategy. Initially, for $t \in [0, t_1^{\text{in}} - 1]$, we put $M(t) = (0,0)$. Next, w.l.o.g. assume that $L(t_1^{\text{out}}) \in \mathrm{S}_5$ — the other cases are defined symmetrically, according to the side type D_5 which $L(t_1^{\text{out}})$ belongs to. By the definition of time moments t_1^{in} and t_1^{out}, we have $t_1^{\text{out}} - 4 \geq t_1^{\text{in}}$, and thus the time interval $(t_1^{\text{in}} - 1, t_1^{\text{out}} + 2]$ may be divided into $(t_1^{\text{in}} - 1, t_1^{\text{in}}]$, $(t_1^{\text{in}}, t_1^{\text{out}} - 4]$, $(t_1^{\text{out}} - 4, t_1^{\text{out}}]$ and $(t_1^{\text{out}}, t_1^{\text{out}} + 2]$. Then, according to this decomposition, the man's trajectory $M(t)$ is defined as follows.

During the time interval $(t_1^{\text{in}} - 1, t_1^{\text{in}}]$, the man moves with speed $V_M = 1$ to the one of the initial points: $(0,1)$,$(-1,0)$, or $(0,1)$, depending on $L(t_1^{\text{in}}) \cap \partial Q_2$ (see Fig. 4(a-b)). Formally, for $t \in (t_1^{\text{in}} - 1, t_1^{\text{in}}]$, we put

$$M(t) = \begin{cases} (0, t - t_1^{\text{in}} + 1) & \text{if } L(t_1^{\text{in}}) \in \mathrm{S}_2; \\ (t_1^{\text{in}} - t - 1, 0) & \text{if } L(t_1^{\text{in}}) \in (\mathrm{E}_2 \cup \mathrm{N}_{22}); \\ (t - t_1^{\text{in}} + 1, 0) & \text{if } L(t_1^{\text{in}}) \in (\mathrm{W}_2 \cup \mathrm{N}_{21}). \end{cases}$$

According to the definition, for all $t \in [0, t_1^{\text{in}}]$, we have $d(L(t), M(t)) \geq 1$. Consider now the time interval $(t_1^{\text{in}}, t_1^{\text{out}} - 4]$. During this interval, the man moves only either horizontally along the x-axis or vertically along the y-axis being

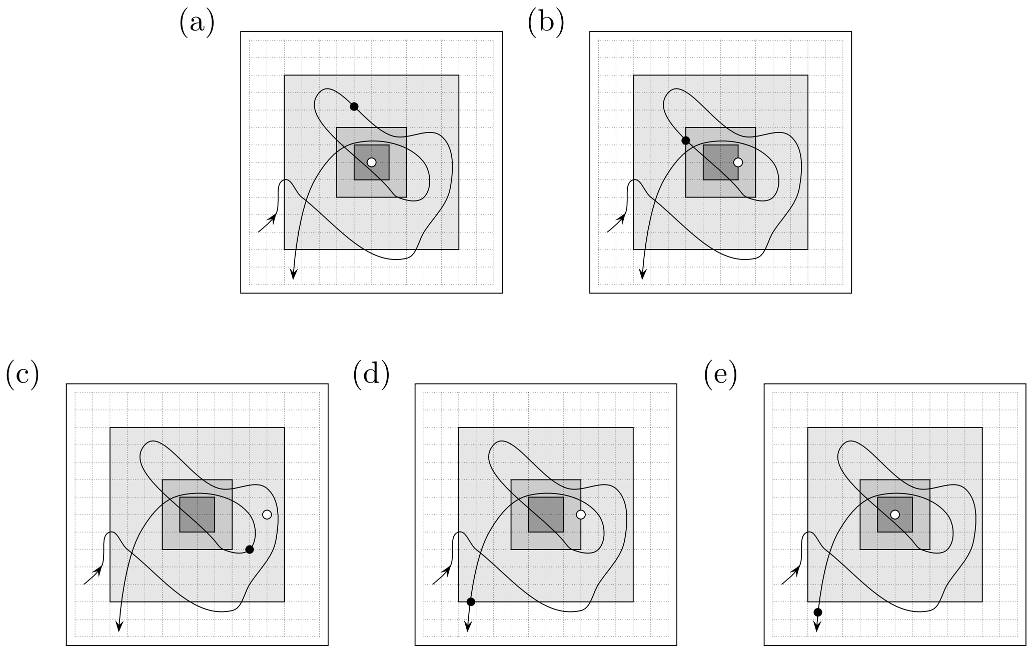


Fig. 4. The case $L(t_j^{\text{in}}) \in \mathrm{W}_2$ and $L(t_j^{\text{out}}) \in \mathrm{S}_5$. (a) Before time $t_j^{\text{in}} - 1$, the man is at the point $(0,0)$. (b) At time $t_j^{\text{in}} - 1$, the man starts moving towards the point $(1,0)$, which is reached at the time moment t_j^{in}. (c) Before time $t_j^{\text{out}} - 4$, the man's x-coordinate is the lion's x-coordinate plus 1. (d) At time $t_j^{\text{out}} - 4$, the man starts moving towards point the $(2,0)$, which is reached at the time moment t_j^{out}. (*e*) At time $t_j^{\text{out}} + 2$, the man is at the point $(0,0)$.

at horizontal (resp. vertical) distance at least one from the current position of the lion (see Fig. 4(c)). Formally, for $t \in (t_1^{\text{in}}, t_1^{\text{out}} - 4]$, we put

$$M(t) = \begin{cases} (0, \max\{1, L_y(t)+1\}) & \text{if } L(t_1^{\text{in}}) \in \mathrm{S}_2; \\ (\min\{-1, L_x(t)-1\}, 0) & \text{if} L(t_1^{\text{in}}) \in (\mathrm{E}_2 \cup \mathrm{N}_{22}); \\ (\max\{1, L_x(t)+1\}, 0) & \text{if } L(t_1^{\text{in}}) \in (\mathrm{W}_2 \cup \mathrm{N}_{21}), \end{cases}$$

where $L_x(t)$ and $L_y(t)$ denote the x and y coordinates of $L(t)$. Note that $d(M(t), L(t)) \geq 1$ for all $t \in (t_1^{\text{in}}, t_1^{\text{out}} - 4]$, and moreover, the function $M(t)$ is continuous on the interval $[0, t_1^{\text{out}} - 4]$. Clearly, the speed of point $M(t)$ is at most the lion's speed V_L.

Consider now the time period $(t_1^{\text{out}} - 4, t_1^{\text{out}}]$. By the definition of $M(t)$ and t_1^{out}, at time $t = t_1^{\text{out}} - 4$:

$$M(t_1^{\text{out}} - 4) = \begin{cases} (0,1) & \text{if } L(t_1^{\text{in}}) \in \mathrm{S}_2; \\ (x,0),\ x \in [-6,-1] & \text{if } L(t_1^{\text{in}}) \in (\mathrm{E}_2 \cup \mathrm{N}_{22}); \\ (x,0),\ x \in [1,6] & \text{if } L(t_1^{\text{in}}) \in (\mathrm{W}_2 \cup \mathrm{N}_{21}), \end{cases}$$

thus either $M(t_1^{\text{out}} - 4)$ is within square Q_2 or it is at distance at most 4 to either point $(-2,0)$ or point $(0,2)$ of square Q_2; of course, $M(t_1^{\text{out}} - 4)$ is on the x-axis. Therefore, when the man starts moving towards square Q_2 at time $t = t_1^{\text{out}} - 4$ with maximum speed, he will reach Q_2 at time t_1^{out} at the latest (see Fig. 4(d)). It remains to prove that with this strategy, $d(M(t), L(t)) \geq 1$ for every $t \in [t_1^{\text{out}} - 4, t_1^{\text{out}}]$. To do this, all we need is notice that for every $t \in [t_1^{\text{out}} - 4, t_1^{\text{out}}]$ we must have $L_y(t) \leq -1$, as long as point $L(t_1^{\text{out}}) \in \mathrm{S}_5$. Hence, any point $p = (a,b)$ with $b = 0$, in particular point $M(t)$, is at the distance at least 1 to $L(t)$ at any time $t \in [t_1^{\text{out}} - 4, t_1^{\text{out}}]$, and thus $d(M(t), L(t)) \geq 1$. Consequently, we can write for $t \in (t_1^{\text{out}} - 4, t_1^{\text{out}}]$:

$$M(t) = \begin{cases} M(t_1^{\text{out}} - 4) & \text{if } L(t_1^{\text{in}}) \in \mathrm{S}_2 \text{ or } M(t_1^{\text{out}} - 4) \in Q_2; \\ M(t_1^{\text{out}} - 4) + [\frac{d(M(t_1^{\text{out}}-4),(-2,0))}{4}(t - t_1^{\text{out}} + 4), 0] & \\ & \text{if } L(t_1^{\text{in}}) \in (\mathrm{E}_2 \cup \mathrm{N}_{22}); \\ M(t_1^{\text{out}} - 4) - [\frac{d(M(t_1^{\text{out}}-4),(2,0))}{4}(t - t_1^{\text{out}} + 4), 0] & \\ & \text{if } L(t_1^{\text{in}}) \in (\mathrm{W}_2 \cup \mathrm{N}_{21}). \end{cases}$$

By the above definition, $M(t) \in \{(0,1)\} \cup [-2,-1] \cup [1,2] \subset Q_2$ at time $t = t_1^{\text{out}}$. Finally, the man moves from its current position $M(t_1^{\text{out}})$ to the point $(0,0)$ as follows:

$$M(t) = \begin{cases} (0, \frac{t_1^{\text{out}} - t}{2} + 1) & \text{if} L(t_1^{\text{in}}) \in \mathrm{S}_2; \\ M(t_1^{\text{out}}) - [\frac{d(M(t_1^{\text{out}}),(0,0))}{4}(t - t_1^{\text{out}}), 0] & \text{if } L(t_1^{\text{in}}) \in (\mathrm{E}_2 \cup \mathrm{N}_{22}); \\ M(t_1^{\text{out}}) + [\frac{d(M(t_1^{\text{out}}),(0,0))}{2}(t - t_1^{\text{out}}), 0] & \text{if} L(t_1^{\text{in}}) \in (\mathrm{W}_2 \cup \mathrm{N}_{21}). \end{cases}$$

Consequently, we get that $M(t) = (0,0)$ at time $t = t_1^{\text{out}} + 2$ (see Fig. 4(e)). And, as for all $t \in (t_1^{\text{out}}, t_1^{\text{out}} + 2]$, point $L(t)$ is in distance at least one to the boundary ∂Q_2 by the definition of time moment t_1^{out}, we get that for all $t \in (t_1^{\text{out}}, t_1^{\text{out}} + 2]$ the lion and the man avoid each other.

Finally, for $t \in (t_1^{\text{out}} + 2, t_2^{\text{in}} - 1]$ we set $M(t) = (0,0)$; notice that $t_2^{\text{in}} - t_1^{\text{out}} \geq 3$ by the definition of time moments t_1^{out} and t_2^{in}. By the definition of $M(t)$, $M(t)$

is continuous on time interval $[0, t_2^{\text{in}} - 1]$, the man's speed is at most one, and $d(M(t), L(t)) \geq 1$. We have thus proved

Claim 8 *For any trajectory $L(t)$ of the lion in Q, there exists a man's trajectory $M(t)$ with the maximum speed at most one such that:*

(1) *at any time $t \in [t_1^{in} - 1, t_2^{in} - 1]$, $d(M(t), L(t)) \geq 1$;*
(2) *$M(t_1^{in} - 1) = M(t_2^{in} - 1) = (0, 0)$.*

By the equality $M(t_1^{\text{in}} - 1) = M(t_2^{\text{in}} - 1) = (0, 0)$, it follows that the same strategy can be applied for any two consecutive time moments t_j^{in} and t_{j+1}^{in}, $j = 2, \ldots, k - 1$. Hence by using a simple invariant argument, we get

Claim 9 *For any trajectory $L(t)$ of the lion in Q, there exists a man's trajectory $M(t)$ with the maximum speed at most one such that:*

(1) *at any time $t \in [0, t_k^{out} + 2]$, $d(M(t), L(t)) \geq 1$;*
(2) *$M(t_1^{in} - 1) = \ldots = M(t_k^{in} - 1) = (0, 0)$.*

The final step for completing the man's strategy is to define $M(t)$ for $t \in (t_k^{\text{out}} + 2, T]$. However, by the definition of time moments t_j^{in}, there is no $t \in (t_k^{\text{out}} + 2, T]$ such that $L(t) \in Q_2$. Thus, if we set $M(t) = M(t_k^{\text{out}} + 2) = (0, 0)$, we get that $\forall_{t \in (t_k^{\text{out}} + 2, T]}\ d(M(t), L(t)) \geq 1$ as well.

Claim 10 *For any trajectory $L(t)$ of the lion in Q, with $L(0)$ and $L(T) \in \partial Q$, there exists a trajectory $M(t)$ of the man such that at any time $t \in [0, T]$, $d(M(t), L(t)) \geq 1$.*

When discussing the case $L(0) \notin \partial Q$ — see the assumption made at the beginning of the section — to determine the strategy $M'(t)$ for the man, (1) we add the shortest segment s connecting $L(t)$ to the boundary ∂Q, (2) we next assume that the lion traverses s with the maximum speed, and (3) put $M'(t) = M(t + |s|)$, where $|s|$ is the length of segment s. Notice that a similar reasoning can be applied when considering the case $L(T) \notin \partial Q$. Consequently, the statement of Theorem 7 follows. □

Finally, notice that the above approach can be adapted for the lion and man problem in which we require a distance of at least r instead of 1. Then all we need is to replace the relevant regions Q_1, Q_2 and Q_5 with regions Q_r, Q_{2r} and Q_{5r}, respectively. This leads to the following corollary.

Corollary 11 *For any trajectory $L(t)$ of the lion in $Q = N \times N$, $N \geq 12r$, there exists a trajectory $M(t)$ of the man such that at any time $t \in [0, T]$, we have $d(M(t), L(t)) \geq r$.*

5 Related problems (in the continuous model)

In this section we consider two other problems: the maximum number of men problem and the spy problem.

5.1 The maximum number of men problem

Let Q_N be an $N \times N$ square and let $L(t) : [0,T] \to Q_N$ be a given trajectory, where $N, T \geq 0$. Let parameter $\zeta(L)$ describe the maximum number of trajectories $M_1, \ldots, M_\zeta(L)$ such that:

(*a*) $\forall_{t\in[0,T]}\ \forall_{i\in\{1,\ldots,\zeta(L)\}}$, we have $d(L(t), M_i(t)) \geq 1$,
(*b*) and $\forall_{t\in[0,T]}\ \forall_{i,j\in\{1,\ldots,\zeta(L)\},i\neq j}$, we have $d(M_i(t), M_j(t)) \geq 1$.

In other words, given the lion's trajectory $L(t)$, we want to find the maximum number of men such that all of them are able to avoid the lion, and moreover, all men avoid each other as well. Define the parameter $\zeta(Q_N)$ as the infimum of $\zeta(L)$ over all lion trajectories: $\zeta(Q_N) = \inf_{L:[0,T]\to Q_N}\ \zeta(L)$, where T is arbitrarily large.

Theorem 12 $\zeta(Q_N) = \Theta(N^2)$

PROOF. We first prove the upper bound on $\zeta(Q_N)$. As for any trajectory $L(t)$ and all $t \in [0,T]$, the open discs $D_L, D_{M_1}, \ldots,\ D_{M_{\zeta(L)}}$ are disjoint — where D_M is the open disc with the radius 1 and the center at point $M(t)$, $M \in \{L, M_1, \ldots, M_{\zeta(L)}\}$, respectively — we clearly have that $\zeta(Q_N) = O(N^2)$.

On the other hand, a careful look at the proof of Theorem 7 shows that the man is able to avoid the lion not only in the case when the man's relevant 'safety' square $\square_M \subseteq Q_N$ is in the middle of square Q_N, but also in the case when the man's relevant safety square $\square_M$ of size 12×12 is just included in Q_N, that is, the center of $\square_M$ is within the distance at least 6 to the boundary ∂Q_N of square Q_N. Therefore, as square Q_N can be decomposed into $\Theta(N^2)$ (interior disjoint) squares of the size 12×12, then we can assign a man per each square of such decomposition (see Fig. 5(a)).

It only remains to show that all the distance conditions are satisfied. This follows from the fact that if the man M_1 is currently avoiding the lion with our strategy within the square $\square_{M_1}$, then the lion is within $\square_{M_1}$ and thus its distance to all other men is at least 1, since all of them except M_1 are just staying at the centers of their relevant squares according to our avoiding strategy. That all men avoid each other follows by a similar argument — notice that even if M_1 intersects the boundary of the safety square of some other man

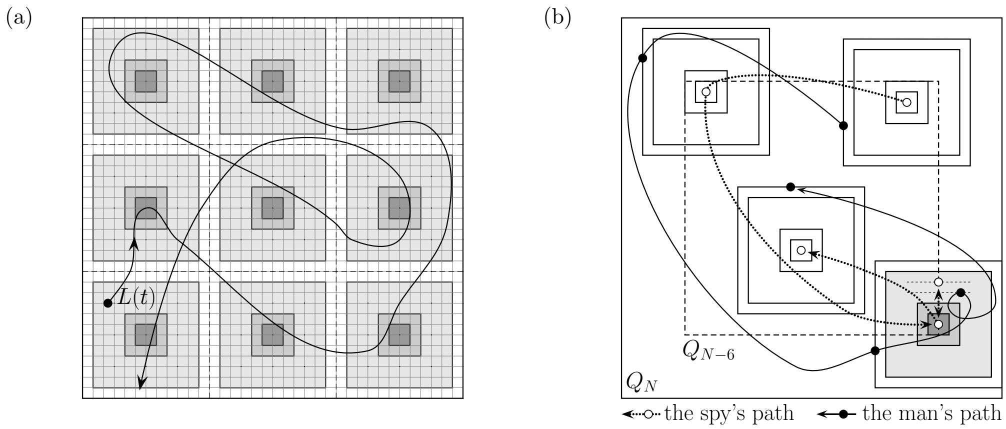


Fig. 5. (a) In the square Q_N of size $N \times N$, there are $(\lfloor \frac{N}{12} \rfloor)^2$ interior disjoint squares of size 12×12; here $N = 36$. (b) An illustration of the strategy for the spy.

M_2, then M_2 is in the center of its own safety square because the lion is in square $\square_{M_1}$. $\square$

5.2 The spy problem

A natural variation of the lion and man problem is the *spy* problem which can be stated as follows. Given a trajectory $M(t)$ and a real $\alpha \geq 1$, find a trajectory $S(t)$ such that at any time $t \in [0, T]$, we have $1 \leq d(M(t), S(t)) \leq \alpha$, that is, the spy S wants to avoid the man M as well as to be not far away from M.

We can use the result in Theorem 7. The idea is illustrated in Fig. 5(b). In square Q_N, with $N \geq 12$, the spy S moves with its safety square $\square_S$ of size 12×12 together with the man in such a manner that the man remains on the boundary $\partial\square_S$ of square $\square_S$ and the shortest distance of the spy to the boundary of square Q_N is at least 6. Clearly, the spy's speed is at most the man's speed and the distance between the man and the spy is at most $6\sqrt{2}$. Suppose now that the spy is not able to move further, that is, he is located on the boundary ∂Q_{N-6}. Then within its safety square $\square_S$, the spy avoids the man according to the strategy described in the proof of Theorem 7 until the spy is again able to move and keep the man on the boundary $\partial\square_S$. A careful analysis of this strategy shows that if the spy avoids the man within square $\square_S$, and the spy is on the boundary ∂Q_{N-6}, then the maximum distance between the spy and the man is $6\sqrt{2}$ (the spy occupies the corner of Q_{N-6}, while the man occupies the relevant corner of Q_N). Consequently, we get

Proposition 13 *For any trajectory $M(t)$ of the man in $Q_N = N \times N$, $N \geq 12$, there exists a trajectory $S(t)$ of the spy such that at any time $t \in [0, T]$, we have $1 \leq d(M(t), S(t)) \leq 6\sqrt{2}$.*

For a given trajectory $M(t) : [0, T] \to Q_N$, let the parameter $\xi(L)$ be defined as follows:

$$\xi(M) = \inf \{\alpha : \exists_{S:[0,T]\to Q_N} \forall_{t\in[0,T]} \; 1 \le d(M(t), S(t)) \le \alpha\}.$$

In other words, given a man's trajectory $M(t)$, we want to find the minimum distance α such that the spy is able to move together with the man within the distance at least 1 and at most α. Define the parameter $\xi(Q_N)$ as the supremum of $\xi(M)$ over all possible man's trajectories, that is, $\xi(Q_N) = \sup_{M:[0,T]\to Q_N} \xi(M)$, where T is arbitrarily large. By the above proposition, we have

Theorem 14 *If $N \ge 12$, then $\xi(Q_N) \le 6\sqrt{2}$.*

6 Final remarks

We think the following variations of the problems considered here may be of interest.

(1) Determine the avoidance number and devise strategies for a man to survive in the case in which the given region is a sphere or a torus. Of course, with a single lion and a single man, the solution becomes trivial.
(2) How many lions can a man avoid in a given region, and what tactics should be applied, if the lions' trajectories are fixed in advance, but the man has no advance knowledge, and hence, he has to compute his trajectory online as the lions' moves are revealed to him?
(3) Consider the variant of the avoidance problem in G_n in which the k given paths are not required to be "continuous": that is, in one step the robot can jump to any grid point in the graph or remain where it is. In addition, let us say that two paths collide when they are at the same grid point at some time instance t, and remove the second condition for collision when traversing the same edge from opposite directions. The reader can verify that the result and argument in our proof still hold for this variant. Letting $j(n)$ to be the maximum number of such "jumping" paths which can be avoided in G_n with a continuous path, we also have $j(n) = \Omega(\sqrt{n})$. We leave for the future the problem of estimating $j(n)$ versus $k(n)$.

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