Convex Polygons in Geometric Triangulations*

Adrian Dumitrescu† Csaba D. Tóth‡

February 14, 2017

Abstract

We show that the maximum number of convex polygons in a triangulation of n points in the plane is $O(1.5029^n)$. This improves an earlier bound of $O(1.6181^n)$ established by van Kreveld, Löffler, and Pach (2012) and almost matches the current best lower bound of $\Omega(1.5028^n)$ due to the same authors. Given a planar straight-line graph G with n vertices, we also show how to compute efficiently the number of convex polygons in G.

Keywords: convex polygon, triangulation, counting, recurrence.

AMS Subject Classification: 05C30, 05C85, 05D99, 57M50.

1 Introduction

Convex polygons. According to the celebrated Erdős-Szekeres theorem [15], every set of n points in the plane, no three on a line, contains $\Omega(\log n)$ points in convex position, and, apart from the constant factor, this bound is the best possible. When the n points are in convex position, then trivially all the $2^n - 1$ nonempty subsets are also in convex position. Erdős [14] proved that the minimum number of subsets in convex position over all n-element point sets with no 3 collinear points, is $\exp(\Theta(\log^2 n))$. See also the survey [20] for many other results related to the Erdős-Szekeres theorem.

Recently, van Kreveld, Löffler, and Pach [18] posed analogous problems concerning the number of convex polygons contained in a triangulation of n points in the plane (as a subgraph); see Fig. 1 (left). A *convex polygon* is a plane straight-line graph cycle whose interior is convex. They proved that the maximum number of convex polygons in a triangulation of n points, no three on a line, is between $\Omega(1.5028^n)$ and $O(1.6181^n)$. Their lower bound comes from a balanced binary triangulation on $2^4 + 1 = 17$ points shown in Fig. 1 (right). At the other end of the spectrum, Dumitrescu et al. [9] showed that the *minimum* number of convex polygons in an n-vertex triangulation is $\Theta(n)$. Here we study the *maximum* number of convex polygons contained in an n-vertex triangulation. Our focus is in the base of the exponent: what is the infimum of $a > 0$ such that every n-vertex triangulation contains $O(a^n)$ convex polygons?

Throughout this paper we consider planar point sets $S \subset \mathbb{R}^2$ in *general position*, in the sense that no 3 points are collinear. A *(geometric) triangulation* of a set $S \subset \mathbb{R}^2$ is a plane straight-line graph with vertex set S such that all bounded faces are triangles that jointly tile the convex hull of S.

*An extended abstract of this paper appeared in the *Proceedings of the 29th International Symposium on Algorithms and Data Structures* (WADS 2015), vol. 9214 of LNCS, Springer, 2015, pp. 289–300.

†Department of Computer Science, University of Wisconsin–Milwaukee, USA. Email: `dumitres@uwm.edu`

‡Department of Mathematics, California State University Northridge, Los Angeles, CA, USA; and Department of Computer Science, Tufts University, Medford, MA, USA. Email: `cdtoth@acm.org`

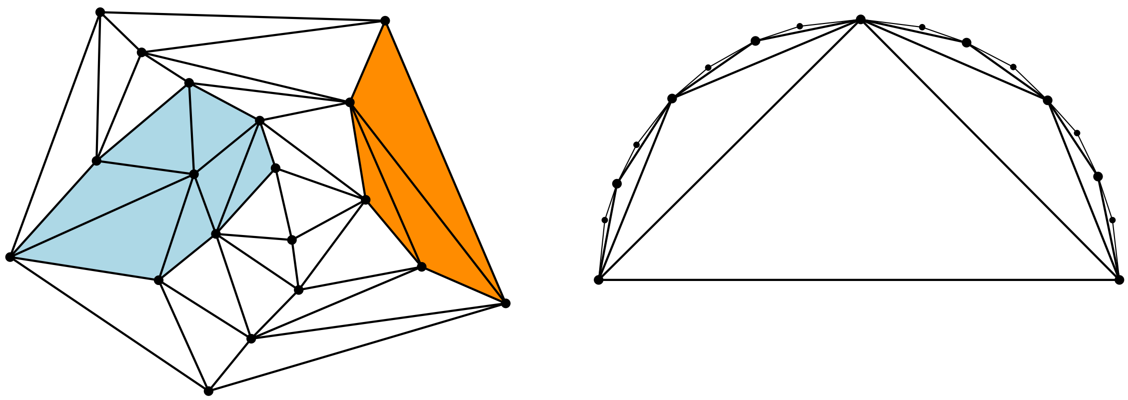

Figure 1: Left: A (geometric) triangulation on 19 points; the boundaries of the two shaded convex polygons are subgraphs of the triangulation. Right: A triangulation on $2^4 + 1 = 17$ points in convex position, whose dual graph is a full binary tree with 8 leaves.

Our results. We first show that the maximum number of convex polygons in an n-vertex triangulation is attained, up to an $O(n)$-factor, for point sets in convex position. Consequently, determining the maximum becomes a purely combinatorial problem. We then prove that the maximum number of convex polygons in a triangulation of n points in the plane is $O(1.5029^n)$. This improves an earlier bound of $O(1.6181^n)$ established by van Kreveld et al. [18] and almost matches the current best lower bound of $\Omega(1.5028^n)$ due to the same authors (Theorem 9 and Corollary 10 in Subsection 2.4). In deriving the new upper bound, we start in Subsection 2.3 with a careful analysis of a balanced binary triangulation illustrated in Fig. 1 (right). In Subsection 2.4 we extend the analysis to *all* triangulations on n points in convex position. In Section 3 we focus on an algorithmic problem: given a planar straight-line graph G with n vertices, determine the number of convex polygons in G. Our main results are summarized in the following.

Theorem 1. *The maximum number of convex polygons in a triangulation of n points in the plane is $O(1.5029^n)$.*

Theorem 2. *Given a planar straight-line graph G with n vertices, the number of convex polygons in G can be computed in $O(n^2)$ time. The convex polygons can be enumerated in an additional $O(1)$-time per edge.*

Related work. In this paper we derive new upper and lower bounds on the maximum number of convex cycles in a straight-line triangulation of n points in the plane. In another article that can be included in the same general theme, Dumitrescu, Mandal, and Tóth [10] recently showed that the (maximum) number of monotone paths in a geometric triangulation of n points in the plane is $O(1.7864^n)$; this improves an earlier upper bound of $O(1.8393^n)$ in [9]; the current best lower bound, $\Omega(1.7003^n)$, appears in [9].

Convex polygons and monotone paths can be defined geometrically—in terms of angles or coordinates. Analogous problems have been previously studied for cycles, spanning cycles, spanning trees, and matchings [6] in n-vertex edge-maximal planar graphs—that are defined in purely graph theoretic terms. For plane straight-line graphs, previous research focused on the maximum number of (noncrossing) configurations such as plane graphs, spanning trees, spanning cycles, triangulations, and others, over all n-element point sets in the plane [1, 2, 11, 16, 21, 23, 24, 25, 26]; see also the two surveys [12, 27]. Early upper bounds in this area were obtained by multiplying the maximum number of triangulations on n point in the plane with the maximum number of desired configurations in an n-vertex triangulation, based on the fact that every planar straight-line graph can be augmented into a triangulation.

The problem of finding the largest convex polygon in a nonconvex container has a long history in computational geometry. Polynomial-time algorithms are known in the plane for the problems of computing a convex polygon with the maximum area or the maximum number of vertices contained in a given simple polygon with n vertices [5, 7, 17] (called the *potato peeling* problem); or spanned by a given set of n points [13].

2 Convex polygons in a triangulation

Section outline. We reduce the problem of determining the maximum number of convex polygons in an n-vertex triangulation (up to polynomial factors) to triangulations of n points in convex position (Theorem 3, Section 2.1). We further reduce the problem to counting convex *paths* between two adjacent hull vertices in a triangulation (Lemma 5, Subsection 2.2). We first analyze the number of convex paths in a balanced binary triangulation, which gives the current best lower bound [18] (Theorem 8, Subsection 2.3). The new insight gained from this analysis is then generalized to derive an upper bound for all n-vertex triangulations (Theorem 9 and Corollary 10, Subsection 2.4).

2.1 Reduction to convex position

For a plane straight-line graph G, let $C(G)$ denote the number of convex polygons in G. For an integer $n \geq 3$, let $C(n)$ be the maximum of $C(G)$ over all plane straight-line graphs G of n points in the plane; and let $C_{\mathrm{x}}(n)$ be the maximum of $C(G)$ over all plane straight-line graphs G of n points *in convex position*. It is clear that $C_{\mathrm{x}}(n) \leq C(n)$ for every $n \geq 3$. The main result of this subsection is the following.

Theorem 3. *For every $n \geq 3$, we have $C(n) \leq (2n-5)\, C_{\mathrm{x}}(n)$.*

Theorem 3 is an immediate consequence of the following lemma.

Lemma 4. *Let T be a triangulation on a set S of n points in the plane, and let f be a (triangular) face of T. Then there exists a set S' of n points in the plane in convex position and a triangulation T' of S' such that the number of convex polygons in T containing f is at most $C(T')$.*

Proof. We construct a point set S' in convex position, a plane straight-line graph G' on S', and then give an injective map from the set of convex polygons in T that contain f into the set of convex polygons of G'. For any triangulation T' of G', we have $C(G') \leq C(T')$.

Let o be a point in the interior of f, not contained in any line determined by S; and let O be a circle centered at o that contains all points in S in its interior; refer to Fig. 2. For each point $p \in S$, let p' be the intersection point of the ray $\overrightarrow{op}$ with O. Let $S' = \{p' : p \in S\}$.

We now construct a plane straight-line graph G' on the point set S'. For two points $p', q' \in S'$, insert an edge $p'q'$ in G' if and only if there is a triangle Δoab whose interior is disjoint from S such that segment ab is contained in an edge of T, point p lies on segment oa, and q lies on ob. Intuitively, the rays $\overrightarrow{op}$ and $\overrightarrow{oq}$ cross a common edge of T at a and b, respectively, and segment ab is "mapped" to $p'q'$.

Note that no two edges in G' cross each other. Indeed, suppose to the contrary that edges $p_1'q_1'$ and $p_2'q_2'$ cross in G'. By construction, there are triangles Δoa_1b_1 and Δoa_2b_2 that induce $p_1'q_1'$ and $p_2'q_2'$, respectively. We may assume without loss of generality that both Δoa_1b_1 and Δoa_2b_2 are oriented counterclockwise, and $\overrightarrow{oa_2}$ enters the interior of Δoa_1b_1 (refer to Fig. 3). Since a_1b_1

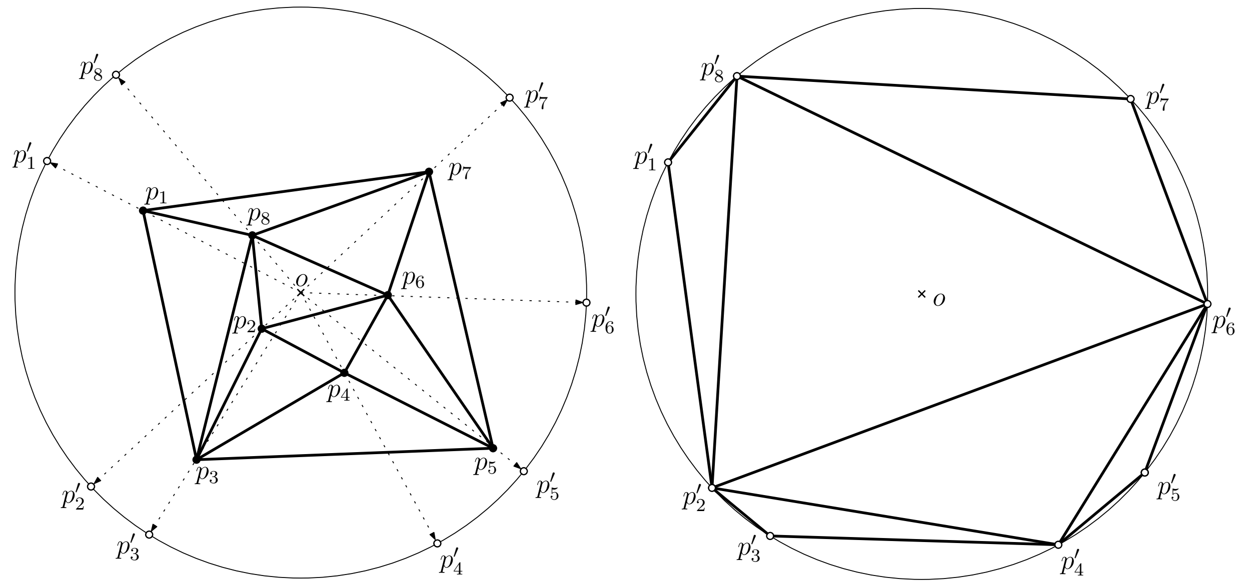

Figure 2: A triangulation T on the point set $\{p_1, \ldots, p_8\}$ (left) is mapped to a triangulation G' on the point set $\{p'_1, \ldots, p'_8\}$ in convex position (right). This induces an injective map from the convex polygons in T containing o to convex polygons in G'. For example, (p_3, p_5, p_6, p_8) is mapped to $(p'_3, p'_4, p'_5, p'_6, p'_8, p'_2)$.

and a_2b_2 do not cross (they may be collinear), segment oa_2 lies in Δoa_1b_1 or segment ob_1 lies in Δoa_2b_2. That is, one of Δoa_1b_1 and Δoa_2b_2 contains a point from S, contradicting the assumption that both triangles are empty.

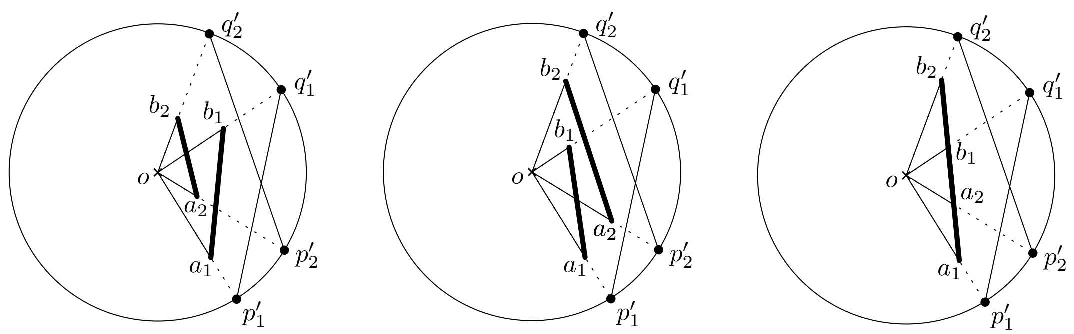

Figure 3: Assume that edges $p'_1q'_1$ and $p'_2q'_2$ cross in G'; both Δoa_1b_1 and Δoa_2b_2 are oriented counterclockwise; and ray oa_2 enters the interior of Δoa_1b_1. Then segment oa_2 lies in Δoa_1b_1 (left), or segment ob_1 lies in Δoa_2b_2 (middle), or both (right).

Finally, we define an injective map from the convex polygons of T that contain o into the convex polygons of G'. To define this map, we first map every edge of T to a path in G'. Let pq be an edge in T, and assume without loss of generality that Δopq is oriented counterclockwise. We map the edge pq to the path $(p' = r'_0, r'_1, \ldots, r'_k, r'_{k+1} = q')$, where $(r_1, \ldots, r_k)$ is the sequence of all points in S lying in the interior of Δopq in counterclockwise order around o. All edges of this path are present in G', since Δor_ir_{i+1} is empty of vertices in S, and both rays $\overrightarrow{or_i}$ and $\overrightarrow{or_{i+1}}$ intersect segment pq, for $i = 0, \ldots, k$. A convex polygon $A = (p_1, \ldots, p_k)$ containing o in T is mapped to the convex polygon A' in G' obtained by concatenating the images of the edges $p_1p_2, \ldots, p_{k-1}p_k$, and p_kp_1. Consequently, the vertex set of A' consists of the images of all points in S that lie on the boundary or in the interior of A.

It remains to show that the above mapping is injective on the convex polygons of T that contain o. Consider a convex polygon $A' = (p'_1, \ldots, p'_k)$ in G' that is the image of some convex polygon in T containing o. Then the preimage A must be a convex polygon in T for which $\{p_1, \ldots, p_k\}$ is the set of points in S that lie on the boundary or in the interior of A. Consequently, A is the boundary of the convex hull of $\{p_1, \ldots, p_k\}$, that is, A' has a unique preimage. □

Proof of Theorem 3. Let T be a (geometric) triangulation with n vertices. Every n-vertex triangulation has at most $2n-4$ faces (including the outer face), and hence at most $2n-5$ bounded faces. By Lemma 4, each bounded face f of T lies in the interior of at most $C_{\rm x}(n)$ convex polygons contained in T. Summing over all bounded faces f, the number of convex polygons in T is bounded by $C(T) \leq (2n-5)\, C_{\rm x}(n)$, as required. □

2.2 Reduction to convex paths

A *convex path* is a *simple* polygonal chain $(p_1, \ldots, p_m)$ that makes a right turn at each interior vertex $p_2, \ldots, p_{m-1}$. Let $P(n)$ denote the maximum number of convex paths between two consecutive hull vertices in a triangulation of n points in convex position. A convex path from a to b is either a direct path consisting of a single segment ab, or a path that can be decomposed into two convex subpaths sharing a common endpoint c, where Δabc is a counterclockwise triangle incident to ab; see Fig. 4.

Figure 4: Convex paths in a triangulation. Left: $P(4) = P(2)\,P(3)+1 = 3$. Right: $P(5) = P(3)\,P(3)+1 = 5$.

Thus $P(n)$ satisfies the following recurrence for $n \geq 3$, with initial values $P(2) = 1$ and $P(3) = 2$.

$$P(n) = \max_{\substack{n_1+n_2=n+1\\ n_1,n_2\geq 2}} \left\{P(n_1)\,P(n_2)+1\right\}. \tag{1}$$

Remark. The values of $P(n)$ for $2 \leq n \leq 18$ are shown in Table 1. It is worth noting that $P(n)$ need not be equal to $P(\lfloor \frac{n+1}{2} \rfloor)\, P(\lceil \frac{n+1}{2} \rceil) + 1$; for instance, $P(7) = P(3)\,P(5) + 1 > P(4)\,P(4) + 1$. That is, the balanced partition of a convex n-gon into two subpolygons does not always maximize $P(n)$. However, we have $P(n) = P(\frac{n+1}{2})\,P(\frac{n+1}{2}) + 1$ for $n = 2^k + 1$ and $k = 1, 2, 3, 4$; these are the values relevant for the (perfectly) balanced binary triangulation discussed in Subsection 2.3.

Let ab be a hull edge of a triangulation T on n points in convex position. Suppose that ab is incident to a counterclockwise triangle Δabc. The edges ac and bc decompose T into three triangulations T_1, Δabc and T_2, of size n_1, 3 and n_2, where $n_1 + n_2 = n + 1$. A convex polygon in T is either (i) contained in T_1; or (ii) contained in T_2; or (iii) the union of ab and a convex path from a to b that passes through c; see Fig. 4. Consequently, $C_{\rm x}(n)$, the maximum number of convex polygons contained in a triangulation of n points in convex position, satisfies the following recurrence:

$$C_{\rm x}(n) = \max_{\substack{n_1+n_2=n+1\\ n_1,n_2\geq 2}} \left\{P(n_1)\,P(n_2) + C_{\rm x}(n_1) + C_{\rm x}(n_2)\right\} \tag{2}$$

for $n \geq 3$, with initial values $C_{\rm x}(2) = 0$ and $C_{\rm x}(3) = 1$. The values of $C_{\rm x}(n)$ for $2 \leq n \leq 9$ are displayed in Table 1.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$P(n)$	1	2	3	5	7	11	16	26	36	56	81	131	183	287	417	677	937
$C_{\mathrm{x}}(n)$	0	1	3	6	11	18	29	45									

Table 1: $P(n)$ and $C_{\mathrm{x}}(n)$ for small n.

Lemma 5. *We have* $C_{\mathrm{x}}(n) \leq \sum_{k=2}^{n-1} P(k)$. *Consequently,* $C_{\mathrm{x}}(n) \leq n\, P(n)$.

Proof. We first prove the inductive inequality:

$$C_{\mathrm{x}}(n) \leq P(n-1) + C_{\mathrm{x}}(n-1). \tag{3}$$

Let T be an arbitrary triangulation of a set S of n points in the plane. Consider the dual graph T^* of T, with a vertex for each triangle in T and an edge for every pair of triangles sharing an edge. It is well known that if the n points are in convex position, then T^* is a tree. Let Δabc be a triangle corresponding to a leaf in T^*, sharing a unique edge, say $e = ab$, with other triangles in T; see Fig. 5.

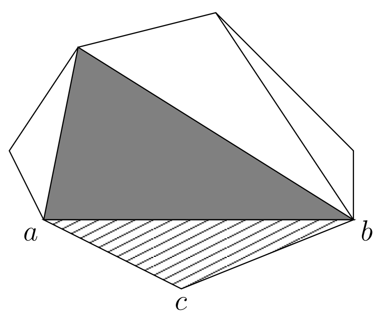

Figure 5: Proof of Lemma 5.

We distinguish two types of convex polygons contained in T: (i) those containing both edges ac and cb, and (ii) those containing neither ac nor cb. Observe that the number of convex polygons of type (i) is at most $P(n-1)$, since any such polygon can be decomposed into the path (b, c, a) and another path connecting a and b in the subgraph of T induced by $S \setminus \{c\}$. Similarly, the number of convex polygons of type (ii) is at most $C_{\mathrm{x}}(n-1)$, since they are contained in the subgraph of T induced by $S \setminus \{c\}$. Altogether we have $C_{\mathrm{x}}(n) \leq P(n-1) + C_{\mathrm{x}}(n-1)$ and (3) is established.

The recursion (3) yields

$$C_{\mathrm{x}}(n) \leq \sum_{k=2}^{n-1} P(k),$$

which is the first inequality in the lemma. Since $P(k) \leq P(k+1)$, for every $k \geq 2$, the second inequality in the lemma follows from the first. □

2.3 The balanced binary triangulation

In this subsection we revisit the balanced binary triangulation on n points, used by van Kreveld, Löffler and Pach [18, Sec. 3.1] in deriving their lower bound $\Omega(1.5028^n)$ on the number of convex polygons contained in a triangulation. For a fixed $k \in \mathbb{N}$, let T_k be the triangulation on 2^k+1 points, say, on a circular arc, such that the dual graph T_k^* is a balanced binary tree; see Fig. 1 (right).

Denote by λ_k the number of convex paths between the leftmost and the rightmost vertex in T_k. As noted in [18], λ_k satisfies the following recurrence:

$$\lambda_{k+1} = \lambda_k^2 + 1, \text{ for } k \geq 0, \qquad \lambda_0 = 1. \tag{4}$$

The values of λ_k for $0 \leq k \leq 5$ are shown in Table 2. Note that $\lambda_k = P(2^k + 1)$ for these values.

k	0	1	2	3	4	5
λ_k	1	2	5	26	677	458330

Table 2: The values of λ_k for small k.

The authors constructed a triangulation of $n = m2^k + 1$ points, for $m \in \mathbb{N}$, by concatenating m copies of T_k along a common circular arc, where consecutive copies share a vertex, and by triangulating the convex hull of the m chords arbitrarily to obtain a triangulation of the n points. The lower bound $\Omega(1.5028^n)$ in [18, Sec. 3.1] is obtained by setting $k = 4$. This construction yields

$$C(n) \geq C_{\mathrm{x}}(n) \geq \lambda_4^{(n-1)/16} = \lambda_4^{-1/16} \left(\lambda_4^{1/16}\right)^n = \Omega(1.5028^n),$$

for every $n = 2^4 m + 1$.

Since the above lower bound directly depends on λ_k, we next examine this dependency. Obviously (4) implies that the sequence $(\lambda_k)^{1/2^k}$ is strictly increasing; as such, $\lambda_k \geq 1.5028^{2^k}$ for every $k \geq 4$. In this subsection (Theorem 8), we establish an almost matching upper bound $\lambda_k \leq 1.50284^{2^k}$, or equivalently, $(\lambda_k)^{1/2^k} \leq 1.50284$ for every $k \geq 0$. The tools used here streamline the way to Subsection 2.4, where we prove an upper bound of $O(1.50285^n)$ on the number of convex polygons contained in any triangulation of n points.

We start by bounding λ_k from above by a product. To this end we frequently use the standard inequality $1 + x \leq \mathrm{e}^x$, where e is the base of the natural logarithm.

Lemma 6. *For $k \in \mathbb{N}$, we have*

$$\lambda_k \leq 2^{2^{k-1}} \prod_{i=1}^{k-1} \left(1 + \frac{1}{2^{2^i}}\right)^{2^{k-1-i}}. \tag{5}$$

Proof. Observe that (4) implies $\lambda_k \geq 2^{2^{k-1}}$ for $k \geq 1$. We thus have

$$\begin{aligned}
\lambda_0 &= 1, \\
\lambda_1 &= 1^2 + 1 = 2, \\
\lambda_2 &= \lambda_1^2 + 1 = 2^2 \left(1 + \frac{1}{2^2}\right), \\
\lambda_3 &= \lambda_2^2 + 1 \leq 2^4 \left(1 + \frac{1}{2^2}\right)^2 \left(1 + \frac{1}{2^4}\right), \\
&\vdots
\end{aligned}$$

We prove (5) by induction on k. The base case $k = 1$ is verified as shown above. For the induction step, we assume that inequality (5) holds for k and show that it holds for $k+1$. Indeed, we have

$$\lambda_{k+1} = \lambda_k^2 + 1 \leq 2^{2^k} \prod_{i=1}^{k-1} \left(1 + \frac{1}{2^{2^i}}\right)^{2^{k-i}} + 1$$
$$\leq 2^{2^k} \prod_{i=1}^{k-1} \left(1 + \frac{1}{2^{2^i}}\right)^{2^{k-i}} \left(1 + \frac{1}{2^{2^k}}\right) = 2^{2^k} \prod_{i=1}^{k} \left(1 + \frac{1}{2^{2^i}}\right)^{2^{k-i}},$$

as required. ☐

The following sequence is instrumental for manipulating the exponents in (5). Let

$$\alpha_k = 2^k + k + 1 \quad \text{for } k \geq 1. \tag{6}$$

That is, $\alpha_1 = 4$, $\alpha_2 = 7$, $\alpha_3 = 12$, $\alpha_4 = 21$, $\alpha_5 = 38$, etc. The way this sequence appears will be evident in Lemma 7, and subsequently, in the chains of inequalities (13) and (14) in the proof of Theorem 9. We next prove the following.

Lemma 7. *For $k \in \mathbb{N}$, we have*

$$\lambda_k \leq 2^{2^{k-1}} \exp\left(2^k \sum_{i=1}^{k-1} 2^{-\alpha_i}\right). \tag{7}$$

Proof. The inequality $1 + x \leq \mathrm{e}^x$ in (5) yields:

$$\lambda_k \leq 2^{2^{k-1}} \prod_{i=1}^{k-1} \left(1 + \frac{1}{2^{2^i}}\right)^{2^{k-1-i}} \leq 2^{2^{k-1}} \exp\left(\sum_{i=1}^{k-1} 2^{k-1-i-2^i}\right)$$
$$= 2^{2^{k-1}} \exp\left(\sum_{i=1}^{k-1} 2^{k-\alpha_i}\right) = 2^{2^{k-1}} \exp\left(2^k \sum_{i=1}^{k-1} 2^{-\alpha_i}\right),$$

as required. ☐

Taking the $1/2^k$ root in (7) yields a first rough approximation (details in Fact 11 of Appendix A):

$$(\lambda_k)^{1/2^k} \leq 2^{2^{k-1}/2^k} \exp\left(2^k/2^k \sum_{i=1}^{k-1} 2^{-\alpha_i}\right) \leq 2^{1/2} \exp\left(\sum_{i=1}^{\infty} 2^{-\alpha_i}\right) \leq 1.5180,$$

To obtain a sharper estimate, we keep the first few terms in the sequence as they are, and only introduce approximations for latter terms.

Theorem 8. *For every $k \in \mathbb{N}$, we have $\lambda_k \leq 1.50284^{2^k}$.*

Proof. From (4), for every $k \geq 0$ we have

$$\lambda_{k+1} = \lambda_k^2 + 1 = \lambda_k^2 \left(1 + \frac{1}{\lambda_k^2}\right) \leq \lambda_k^2 \left(1 + \frac{1}{2^{2^k}}\right),$$
$$\lambda_{k+2} = \lambda_{k+1}^2 + 1 = \lambda_{k+1}^2 \left(1 + \frac{1}{\lambda_{k+1}^2}\right) \leq \lambda_k^4 \left(1 + \frac{1}{2^{2^k}}\right)^2 \left(1 + \frac{1}{2^{2^{k+1}}}\right),$$
$$\lambda_{k+3} = \lambda_{k+2}^2 + 1 = \lambda_{k+2}^2 \left(1 + \frac{1}{\lambda_{k+2}^2}\right) \leq \lambda_k^8 \left(1 + \frac{1}{2^{2^k}}\right)^4 \left(1 + \frac{1}{2^{2^{k+1}}}\right)^2 \left(1 + \frac{1}{2^{2^{k+2}}}\right),$$
$$\vdots$$

For every $k \geq 0$ and $i \geq 1$ we have

$$\lambda_{k+i} = \lambda_{k+i-1}^2 + 1 = \lambda_{k+i-1}^2 \left(1 + \frac{1}{\lambda_{k+i-1}^2}\right) \leq (\lambda_k)^{2^i} \prod_{j=1}^{i} \left(1 + \frac{1}{2^{2^{k+j-1}}}\right)^{2^{i-j}}$$
$$\leq (\lambda_k)^{2^i} \exp\left(\sum_{j=1}^{i} 2^{i+k-\alpha_{k+j-1}}\right) = (\lambda_k)^{2^i} \exp\left(2^{i+k} \sum_{j=1}^{i} 2^{-\alpha_{k+j-1}}\right).$$

Consequently,

$$(\lambda_{k+i})^{1/2^{k+i}} \leq (\lambda_k)^{2^i/2^{i+k}} \exp\left(\sum_{j=1}^{i} 2^{-\alpha_{k+j-1}}\right) = (\lambda_k)^{1/2^k} \exp\left(\sum_{j=1}^{i} 2^{-\alpha_{k+j-1}}\right).$$

For $i \geq 1$ and $k = 4$ the above inequality is

$$(\lambda_{4+i})^{1/2^{4+i}} \leq (\lambda_4)^{1/2^4} \exp\left(\sum_{j=1}^{i} 2^{-\alpha_{4+j-1}}\right) = (\lambda_4)^{1/2^4} \exp\left(\sum_{j=4}^{i+3} 2^{-\alpha_j}\right).$$

Put $k = i + 4 \geq 5$; and so the following holds for $k \geq 5$:

$$(\lambda_k)^{1/2^k} \leq (\lambda_4)^{1/2^4} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right) = 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right)$$
$$\leq 677^{1/16} \exp\left(\sum_{i=4}^{\infty} 2^{-\alpha_i}\right) \leq 1.50284. \tag{8}$$

The last inequality in the above chain is Fact 12 in Appendix A. The inequality $(\lambda_k)^{1/2^k} \leq 1.50284$ also holds for $0 \leq k \leq 4$, and thus for all $k \geq 0$, as required (recall that the sequence $(\lambda_k)^{1/2^k}$ is strictly increasing). □

2.4 Convex paths in a triangulation of a convex point set

In this subsection we show that the maximum number of convex paths between two adjacent vertices in a triangulation of n points in convex position is $O(1.50284^n)$, that is, $P(n) = O(1.50284^n)$. In the main step, a complex proof by induction yields the following.

Theorem 9. *Let $n \geq 2$, where $2^k + 1 \leq n \leq 2^{k+1}$. Then*

$$P(n)^{\frac{1}{n-1}} \leq (P(17))^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right) = 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right). \tag{9}$$

Proof. We prove the inequality by induction on n. The base cases $2 \leq n \leq 32$ are satisfied; this is verified by direct calculation in Facts 13 and 14 of Appendix A:

$$\max_{2 \leq n \leq 16} P(n)^{\frac{1}{n-1}} = P(9)^{1/8} = 26^{1/8} = 1.50269\ldots.$$
$$\max_{17 \leq n \leq 32} P(n)^{\frac{1}{n-1}} = P(17)^{1/16} = 677^{1/16} = 1.50283\ldots.$$

Since $0 \le k \le 4$, it follows that

$$\max_{2\le n\le 32} P(n)^{\frac{1}{n-1}} = P(17)^{1/16} = 677^{1/16} = 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right).$$

Assume now that $n \ge 33$, hence $k \ge 5$, and that the required inequality holds for all smaller values. We will show that for all pairs $n_1, n_2 \ge 2$ with $n_1+n_2 = n+1$, the expression $P(n_1)\,P(n_2)+1$ is bounded from above as required. Note that since $n_1 + n_2 = n + 1$, we have $n_1, n_2 \le n - 1$, so using the induction hypothesis for n_1 and n_2 is justified. It suffices to consider pairs with $n_1 \le n_2$. We distinguish two cases:

Case 1: $2 \le n_1 \le 16$. Since $n \ge 33$, we have $18 \le n_2 \le n - 1$. By the induction hypothesis we have

$$P(n_2)^{1/(n_2-1)} \le 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right).$$

Further,

$$\begin{aligned} P(n) &\le P(n_1)\,P(n_2) + 1 \\ &\le P(n_1)\,677^{\frac{n_2-1}{16}} \exp\left((n_2-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right) + 1 \\ &\le P(n_1)\,677^{\frac{n_2-1}{16}} \exp\left((n_2-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right)\left(1 + (P(n_1))^{-1}\,677^{-\frac{n_2-1}{16}}\right) \\ &\le P(n_1)\,677^{\frac{n_2-1}{16}} \exp\left((n_2-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right)\exp\left((P(n_1))^{-1}\,677^{-\frac{n_2-1}{16}}\right). \end{aligned}$$

To settle Case 1, it suffices to show that

$$P(n_1)\,677^{\frac{n_2-1}{16}} \exp\left((n_2-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right)\exp\left((P(n_1))^{-1}\,677^{-\frac{n_2-1}{16}}\right) \le$$

$$\le 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right),$$

or equivalently,

$$P(n_1)\,\exp\left((P(n_1))^{-1}\,677^{-\frac{n_2-1}{16}}\right) \le 677^{\frac{n_1-1}{16}} \exp\left((n_1-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right). \tag{10}$$

We have $n_1 + n_2 = n + 1$, hence $n_2 - 1 = n - n_1 \ge 33 - n_1$. By Fact 15 in Appendix A, the following inequality holds for $2 \le n_1 \le 16$:

$$P(n_1)\,\exp\left((P(n_1))^{-1}\,677^{-\frac{33-n_1}{16}}\right) \le 677^{\frac{n_1-1}{16}}. \tag{11}$$

Now (11) in conjunction with $n_2 - 1 \ge 33 - n_1$ yields

$$\begin{aligned} P(n_1)\,\exp\left((P(n_1))^{-1}\,677^{-\frac{n_2-1}{16}}\right) &\le P(n_1)\,\exp\left((P(n_1))^{-1}\,677^{-\frac{33-n_1}{16}}\right) \\ &\le 677^{\frac{n_1-1}{16}} \le 677^{\frac{n_1-1}{16}} \exp\left((n_1-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right), \end{aligned}$$

as required by (10).

Case 2: $n_1 \ge 17$. We distinguish two subcases, $n \le 2^k + 2$ and $n \ge 2^k + 3$.

Case 2.a: $n \le 2^k + 2$. Since $n_1 \ge 17 \ge 3$ it follows that $n_2 \le 2^k$ and thus the inductive upper bound on $P(n_2)^{\frac{1}{n_2-1}}$ has a shorter expansion (up to $k-2$):

$$P(n_2)^{\frac{1}{n_2-1}} \le 677^{1/16} \exp\left(\sum_{i=4}^{k-2} 2^{-\alpha_i}\right), \text{ or equivalently,}$$

$$P(n_2) \le 677^{\frac{n_2-1}{16}} \exp\left((n_2-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right).$$

Since $n_1 \le n_2$, the same holds for $P(n_1)^{\frac{1}{n_1-1}}$:

$$P(n_1)^{\frac{1}{n_1-1}} \le 677^{1/16} \exp\left(\sum_{i=4}^{k-2} 2^{-\alpha_i}\right), \text{ or equivalently,}$$

$$P(n_1) \le 677^{\frac{n_1-1}{16}} \exp\left((n_1-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right).$$

Since $n_1 + n_2 = n + 1$, putting these two inequalities together yields:

$$\begin{aligned} P(n_1)P(n_2) + 1 &\le 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right) + 1 \\ &\le 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right)\left(1 + 677^{-\frac{n-1}{16}}\right) \\ &\le 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right)\exp\left(677^{-\frac{n-1}{16}}\right). \end{aligned}$$

To settle Case 2.a, it suffices to show the following.

$$677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right)\exp\left(677^{-\frac{n-1}{16}}\right) \le 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right). \tag{12}$$

Recall that $k \ge 5$ and this inequality is needed here; for $k \le 4$, the second factors on the left and the right side of (12) are both equal to 1, and so (12) would not hold. Note that (12) is equivalent to

$$\begin{aligned} \exp\left((n-1)\sum_{i=4}^{k-2} 2^{-\alpha_i}\right)\exp\left(677^{-\frac{n-1}{16}}\right) &\le \exp\left((n-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right), \\ \exp\left(677^{-\frac{n-1}{16}}\right) &\le \exp\left((n-1)2^{-\alpha_{k-1}}\right), \\ 677^{-\frac{n-1}{16}} &\le (n-1)2^{-\alpha_{k-1}}. \end{aligned}$$

Recall that $\alpha_{k-1} = 2^{k-1} + k$; we also have $n - 1 \ge 2^k$, hence $\frac{n-1}{2} \ge 2^{k-1}$. These relations yield

$$(n-1)2^{-\alpha_{k-1}} = \frac{n-1}{2^{\alpha_{k-1}}} \ge \frac{2^k}{2^{\alpha_{k-1}}} = \frac{1}{2^{2^{k-1}}} \ge \frac{1}{2^{\frac{n-1}{2}}} \ge \frac{1}{677^{\frac{n-1}{16}}}, \tag{13}$$

as required.

Case 2.b: $n \geq 2^k + 3$. Assume that $2^{k_1} + 1 \leq n_1 \leq 2^{k_1+1}$ for a suitable $4 \leq k_1 \leq k$; indeed, $n_1 \geq 17$ implies $k_1 \geq 4$. If we would have $k_1 = k$ then $n_2 \geq n_1 \geq 2^k + 1$ hence $n_1 + n_2 \geq 2^{k+1} + 2$, or $n \geq 2^{k+1} + 1$, in contradiction to the original assumption on n in the theorem. It follows that $k_1 \leq k - 1$, and further that $n_1 \leq 2^{k_1+1} \leq 2^k$ and $n \geq 2^{k_1+1} + 3$. The inductive upper bound on $P(n_1)^{\frac{1}{n_1-1}}$ has the expansion:

$$P(n_1)^{\frac{1}{n_1-1}} \leq 677^{1/16} \exp\left(\sum_{i=4}^{k_1-1} 2^{-\alpha_i}\right), \text{ or equivalently,}$$

$$P(n_1) \leq 677^{\frac{n_1-1}{16}} \exp\left((n_1 - 1)\sum_{i=4}^{k_1-1} 2^{-\alpha_i}\right).$$

By the inductive assumption we also have

$$P(n_2)^{\frac{1}{n_2-1}} \leq 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right), \text{ or equivalently,}$$

$$P(n_2) \leq 677^{\frac{n_2-1}{16}} \exp\left((n_2 - 1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right).$$

Recall that $k_1 \geq 4$. Since $n_1 + n_2 = n + 1$, putting these two inequalities together yields:

$$\begin{aligned}
P(n_1)P(n_2) + 1 &\leq 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k_1-1} 2^{-\alpha_i} + (n_2 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right) + 1 \\
&\leq 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k_1-1} 2^{-\alpha_i} + (n_2 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right)\left(1 + 677^{-\frac{n-1}{16}}\right) \\
&\leq 677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k_1-1} 2^{-\alpha_i} + (n_2 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right) \exp\left(677^{-\frac{n-1}{16}}\right).
\end{aligned}$$

To settle Case 2.b, it suffices to show that

$$677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k_1-1} 2^{-\alpha_i} + (n_2 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right) \exp\left(677^{-\frac{n-1}{16}}\right) \leq$$
$$677^{\frac{n-1}{16}} \exp\left((n-1)\sum_{i=4}^{k-1} 2^{-\alpha_i}\right),$$

or equivalently,

$$\begin{aligned}
\exp\left((n_2 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right) \exp\left(677^{-\frac{n-1}{16}}\right) &\leq \exp\left((n-1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right), \\
\exp\left(677^{-\frac{n-1}{16}}\right) &\leq \exp\left((n - n_2)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}\right), \\
677^{-\frac{n-1}{16}} &\leq (n_1 - 1)\sum_{i=k_1}^{k-1} 2^{-\alpha_i}.
\end{aligned}$$

Recall that $\alpha_{k_1} = 2^{k_1} + k_1 + 1$ and that $n_1 - 1 \geq 2^{k_1}$ by the assumption of Case 2.b; we also have (by the same reasons):

$$n \geq 2^{k_1+1} + 3 \quad \Rightarrow \quad \frac{n-1}{2} \geq 2^{k_1} + 1,$$
$$k_1 \leq k - 1 \quad \Rightarrow \quad 2^{-\alpha_{k_1}} \leq \sum_{i=k_1}^{k-1} 2^{-\alpha_i}.$$

From these relations we deduce that

$$\frac{1}{677^{\frac{n-1}{16}}} \leq \frac{1}{2^{\frac{n-1}{2}}} \leq \frac{1}{2^{2^{k_1}+1}} = \frac{2^{k_1}}{2^{\alpha_{k_1}}} \leq \frac{n_1 - 1}{2^{\alpha_{k_1}}} \leq (n_1 - 1) \sum_{i=k_1}^{k-1} 2^{-\alpha_i}, \tag{14}$$

as required. □

Corollary 10. $P(n) = O(1.50284^n)$.

Proof. By Theorem 9 and Fact 12 in Appendix A) we obtain

$$P(n)^{\frac{1}{n}} \leq P(n)^{\frac{1}{n-1}} \leq 677^{1/16} \exp\left(\sum_{i=4}^{k-1} 2^{-\alpha_i}\right) \leq 677^{1/16} \exp\left(\sum_{i=4}^{\infty} 2^{-\alpha_i}\right) \leq 1.50284.$$ □

Proof of Theorem 1. By Corollary 10 we have $C_{\mathrm{x}}(n) = O(nP(n))$ and by Theorem 3 we have $C(n) \leq (2n-5)\, C_{\mathrm{x}}(n)$. It follows that

$$C(n) \leq (2n-5)\, C_{\mathrm{x}}(n) \leq 2n^2\, P(n) \leq 2n^2 \cdot 1.50284^n = O(1.50285^n),$$

as required. □

3 Counting algorithm

The number of crossing-free structures (matchings, spanning trees, spanning cycles, triangulations) on a set of n points in the plane is known to be exponential in n [11, 16, 21, 24, 25, 26]. It is a challenging problem to determine the number of configurations faster than listing all such configurations (i.e., count faster than enumerate) [3]. Exponential-time algorithms have been developed for triangulations [4], planar graphs [22], and matchings [28] that count these structures exponentially faster than the number of structures. Recently, it has been shown that the number of triangulations on n points in the plane can be counted in subexponential time [19], and this result extends to counting noncrossing prefect matchings, spanning trees, spanning cycles, 3-regular graphs, and more.

Here we show that given a plane straight-line graph G with n vertices (e.g., a triangulation), convex polygons in G can be counted in polynomial time.

Proof of Theorem 2. Let $G = (V, E)$ be a plane straight line graph. For counting and enumerating convex cycles (i.e., polygons) in G, we adapt a dynamic programming approach by Eppstein et al. [13], originally developed for finding subsets in convex position of an n-element point set in the plane optimizing various parameters, e.g., the area or the perimeter of the convex hull. The dynamic program relies on the following observations:

1. Assume, by rotating G if necessary, that no two vertices have the same x- or y-coordinates. Denote the vertices of G by $V = \{v_1, \ldots, v_n\}$, ordered by their x-coordinates. Every convex polygon ξ has a leftmost vertex v_i and a rightmost vertex v_k. The points v_i and v_k decompose ξ into two x-monotone chains in G: a convex (lower) chain and a concave (upper) chain connecting v_i and v_k; refer to Fig. 6.

2. Conversely, the union of any convex and concave chains between vertices v_i and v_k form a convex polygon, unless the two chains are equal, which happens when both chains are the one-edge chain $v_i v_k$. For $1 \le i < k \le n$, denote by $\texttt{cup}(i,k)$ and $\texttt{cap}(i,k)$, respectively, the number of x-monotone convex and concave chains between v_i and v_k.

3. The number of convex polygons in G whose leftmost and rightmost vertex, respectively, are v_i and v_k is $\texttt{cup}(i,k) \cdot \texttt{cap}(i,k)$ if $v_i v_k \notin E$, and $\texttt{cup}(i,k) \cdot \texttt{cap}(i,k) - 1$ otherwise. Consequently, it is enough to compute $\texttt{cup}(i,k)$ and $\texttt{cap}(i,k)$ for all $1 \le i < k \le n$ in $O(n^2)$ time. We consider $\texttt{cup}(i,k)$ only, the case of $\texttt{cap}(i,k)$ is analogous.

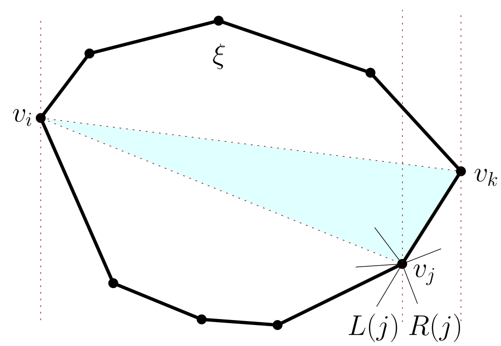

Figure 6: A convex polygon where v_i is the leftmost vertex, v_k is the rightmost vertex and v_j is the penultimate vertex on the lower (convex) chain between v_i and v_k. The edges in $L(j)$ and $R(j)$, whose right and left endpoint, respectively, is v_j are shown partially in the neighborhood of v_j.

We introduce a third parameter. For $1 \le i \le j < k \le n$ where $v_j v_k \in E$, let $\texttt{cup}(i,j,k)$ be the number of x-monotone convex chain between v_i and v_k whose rightmost edge is $v_j v_k$. Since G is planar, the number of triples $1 \le i \le j < k \le n$, where $v_j v_k \in E$, is bounded from above by $|V| \cdot |E| = O(n^2)$. Further, for each vertex $v_j \in V$, we partition the incident edges into two subsets: let $L(j)$ be the set of indices ℓ such that $v_\ell v_j \in E$ and v_ℓ is the *left* endpoint of $v_\ell v_j$; and let $R(j)$ be the set of indices k such that $v_j v_k \in E$ and v_k is the *right* endpoint of $v_j v_k$. Note that for all $1 \le i < k \le n$ we have

$$\texttt{cup}(i,k) = \sum_{j \in L(k): i \le j} \texttt{cup}(i,j,k). \tag{15}$$

We compute the values $\texttt{cup}(i,j,k)$ in $O(n^2)$ time by dynamic programming; the entries are computed in increasing order of $j - i$. In a preprocessing step, we sort the indices $\ell \in L(j)$ by $\texttt{slope}(v_\ell v_j)$, and analogously the indices $k \in R(j)$ by $\texttt{slope}(v_j v_k)$, for all $j = 1, \ldots, n$. This takes $O(\sum_{j=1}^n \deg(v_j) \log \deg(v_j)) = O(\sum_{j=1}^n \deg(v_j) \log n) = O(n \log n)$ time.

Fix an index i, $1 \le i \le n-1$. For every $j \in \{i, \ldots, n-1\}$ and $k \in R(j)$, we compute $\texttt{cup}(i,j,k)$ in $O(n)$ time in two nested loops. For $j = i$ and all $k \in R(j)$, put $\texttt{cup}(i,j,k) = 1$. Consider next the values $j = i+1, \ldots, n$: if an x-monotone convex chain between v_i and v_k contains the edge $v_j v_k$, and its penultimate edge is $v_\ell v_j$, then $\texttt{slope}(v_\ell v_j) < \texttt{slope}(v_j v_k)$. Consequently, for every

$k \in R(j)$, we have

$$\texttt{cup}(i,j,k) = \sum_{\substack{\ell \in L(j)\,:\, i \le \ell \text{ and} \\ \texttt{slope}(v_\ell v_j) < \texttt{slope}(v_j v_k)}} \texttt{cup}(i,\ell,j). \tag{16}$$

We can compute $\texttt{cup}(i,j,k)$ for all $k \in R(j)$ using (16), since $\texttt{cup}(i,\ell,j)$ for all $\ell = i, \ldots, j-1$ has already been computed. If $k, k' \in R(j)$ and $\texttt{slope}(v_j v_k) < \texttt{slope}(v_j v_{k'})$, then

$$\texttt{cup}(i,j,k') = \texttt{cup}(i,j,k) + \sum_{\substack{\ell \in L(j)\,:\, i \le \ell \text{ and} \\ \texttt{slope}(v_j v_k < \texttt{slope}(v_\ell v_j) < \texttt{slope}(v_j v_{k'})}} \texttt{cup}(i,\ell,j). \tag{17}$$

Since $L(j)$ and $R(j)$ are ordered by $\texttt{slope}(v_\ell v_j)$ and $\texttt{slope}(v_j v_k)$, respectively, we can use (17) to compute $\texttt{cup}(i,j,k)$ for all $k \in R(j)$ in this order in $O(\deg(v_j))$ time. The running time for a fixed index i, $1 \le i \le n-1$, is $O(\sum_{j=i}^{n-1} \deg(v_j)) = O(\sum_{j=1}^{n} \deg(v_j)) = O(2|E|) = O(n)$. The overall running time, over all $i = 1, \ldots, n-1$, is $O(n^2)$, as claimed.

To enumerate all convex polygons in G, we compute the *set* of convex and concave chains corresponding to the values $\texttt{cup}(i,k)$ and $\texttt{cap}(i,k)$ for all pairs (i,k), $1 \le i < k \le n$, where $\texttt{cup}(i,k) \cdot \texttt{cap}(i,k) \ne 0$ (the remaining pairs do not contribute any convex polygons). For any such pair (i,k), we enumerate all convex polygons with leftmost vertex v_i and rightmost vertex v_k by reporting all combinations of x-monotone convex chains and x-monotone concave chains between v_i and v_k, except for the possible combination of two single-edge chains when $v_i v_k \in E$.

To this end, we first prune the recursion tree on $\texttt{cup}(i,j,k)$ (resp., $\texttt{cap}(i,j,k)$) induced by the recursion formula (17). Construct both recursion trees (for $\texttt{cup}(i,j,k)$ and $\texttt{cap}(i,j,k)$). In a top-down traversal, mark all nodes that contribute a nonzero value in the recursive computation of $\texttt{cup}(i,k)$ and $\texttt{cap}(i,k)$, where $\texttt{cup}(i,k) \cdot \texttt{cap}(i,k) \ne 0$. For these pairs (i,k), $1 \le i < k \le n$, we then compute the corresponding *set* of convex (resp., concave) chains by tracing back the recursion tree and concatenating edges one-by-one in $O(1)$ time per edge (cf. [8, p. 387]). □

Acknowledgment. The authors are grateful to an anonymous reviewer for a very careful reading of the manuscript and for pertinent remarks. In particular, the improvements suggested by the reviewer concerning the counting algorithm in Section 3 were received with great appreciation.

References

[1] O. Aichholzer, T. Hackl, B. Vogtenhuber, C. Huemer, F. Hurtado, and H. Krasser, On the number of plane geometric graphs, *Graphs and Combinatorics* **23(1)** (2007), 67–84.

[2] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free subgraphs, *Annals of Discrete Mathematics* **12** (1982), 9–12.

[3] V. Alvarez, K. Bringmann, S. Ray, and R. Seidel, Counting triangulations and other crossing-free structures approximately, *Computational Geometry: Theory & Applications* **48(5)** (2015), 386–397.

[4] V. Alvarez and R. Seidel, A simple aggregative algorithm for counting triangulations of planar point sets and related problems, in *Proc. 29th Sympos.on Comput. Geom.* (SOCG), ACM Press, 2013, pp. 1–8.

[5] B. Aronov, M. van Kreveld, M. Löffler, and R. I. Silveira, Peeling meshed potatoes, *Algorithmica* **60(2)** (2011), 349–367.

[6] K. Buchin, C. Knauer, K. Kriegel, A. Schulz, and R. Seidel. On the number of cycles in planar graphs, in *Proc. 13th Annual International Conference on Computing and Combinatorics* (COCOON), LNCS 4598, Springer, 2007, pp. 97–107.

[7] J. S. Chang and C. K. Yap, A polynomial solution for the potato-peeling problem, *Discrete & Computational Geometry* **1** (1986), 155–182.

[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 3rd edition, MIT Press, Cambridge, 2009.

[9] A. Dumitrescu, M. Löffler, A. Schulz, and Cs. D. Tóth, Counting carambolas, *Graphs and Combinatorics* **32(3)** (2016), 923–942.

[10] A. Dumitrescu, R. Mandal, and Cs. D. Tóth, Monotone paths in geometric triangulations, Preprint, arXiv:1608.04812, 2016. An extended abstract of an earlier version in *Proc. 27th International Workshop on Combinatorial Algorithms (IWOCA 2016)*, LNCS 9843, pp. 411–422, Springer, 2016.

[11] A. Dumitrescu, A. Schulz, A. Sheffer, and Cs. D. Tóth, Bounds on the maximum multiplicity of some common geometric graphs, *SIAM Journal on Discrete Mathematics* **27(2)** (2013), 802–826.

[12] A. Dumitrescu and Cs. D. Tóth, Computational Geometry Column 54, *SIGACT News Bulletin* **43(4)** (2012), 90–97.

[13] D. Eppstein, M. Overmars, G. Rote, and G. Woeginger, Finding minimum area k-gons, *Discrete & Computational Geometry* **7(1)** (1992), 45–58.

[14] P. Erdős, Some more problems on elementary geometry, *Gazette of the Australian Mathematical Society* **5** (1978), 52–54.

[15] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.

[16] A. García, M. Noy and A. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N, *Computational Geometry: Theory & Applications* **16(4)** (2000), 211–221.

[17] J. E. Goodman, On the largest convex polygon contained in a non-convex n-gon or how to peel a potato, *Geometria Dedicata* **11** (1981), 99–106.

[18] M. J. van Kreveld, M. Löffler, and J. Pach, How many potatoes are in a mesh?, in *Proc. 23rd Internat. Sympos. on Alg. Comput.* (ISAAC), LNCS 7676, Springer, 2012, pp. 166–176.

[19] D. Marx and T. Miltzow, Peeling and nibbling the cactus: Subexponential-time algorithms for counting triangulations and related problems, in *Proc. 32nd International Symposium on Computational Geometry* (SoCG), LIPIcs 51, Schloss Dagstuhl, 2016, article 52.

[20] W. Morris and V. Soltan, The Erdős-Szekeres problem on points in convex position—a survey, *Bulletin of AMS* **37** (2000), 437–458.

[21] A. Razen, J. Snoeyink, and E. Welzl, Number of crossing-free geometric graphs vs. triangulations, *Electronic Notes in Discrete Mathematics* **31** (2008), 195–200.

[22] A. Razen and E. Welzl, Counting plane graphs with exponential speed-up, in *Rainbow of Computer Science*, Springer, 2011, pp. 36–46.

[23] M. Sharir and A. Sheffer, Counting triangulations of planar point sets, *The Electronic Journal of Combinatorics* **18** (2011), P70.

[24] M. Sharir and A. Sheffer, Counting plane graphs: cross-graph charging schemes, *Combinatorics, Probability & Computing* **22(6)** (2013), 935–954.

[25] M. Sharir, A. Sheffer, and E. Welzl, Counting plane graphs: perfect matchings, spanning cycles, and Kasteleyn's technique, *Journal of Combinatorial Theory, Ser. A* **120(4)** (2013), 777–794.

[26] M. Sharir and E. Welzl, On the number of crossing-free matchings, cycles, and partitions, *SIAM Journal on Computing* **36(3)** (2006), 695–720.

[27] A. Sheffer, Numbers of plane graphs, https://adamsheffer.wordpress.com/numbers-of-plane-graphs/ (version of May, 2015).

[28] M. Wettstein, Counting and enumerating crossing-free geometric graphs, in *Proc. 30th Sympos. on Comput. Geom.* (SOCG), ACM Press, 2014, pp. 1–10. Full paper available at arXiv:1604.05350, 2016.

A Numeric calculations

We need the following numerical estimates.

Fact 11. *The following inequality holds:*

$$2^{1/2}\exp\left(\sum_{i=1}^{\infty}2^{-\alpha_i}\right)\le 1.5180.$$

Proof. An easy calculation yields an upper bound on the sum $\sum_{i=1}^{\infty}2^{-\alpha_i}$:

$$\begin{aligned}\sum_{i=1}^{\infty}2^{-\alpha_i}&=2^{-4}+2^{-7}+2^{-12}+2^{-21}+\dots\\&\le 2^{-4}+2^{-7}+\sum_{i=1}^{\infty}2^{-11-i}=2^{-4}+2^{-7}+2^{-11}.\end{aligned}$$

It follows that

$$2^{1/2}\exp\left(\sum_{i=1}^{\infty}2^{-\alpha_i}\right)\le 2^{1/2}\exp\left(2^{-4}+2^{-7}+2^{-11}\right)\le 1.5180,$$

as required. □

Fact 12. *The following inequality holds:*

$$677^{1/16}\exp\left(\sum_{i=4}^{\infty}2^{-\alpha_i}\right)\le 1.50284.$$

Proof. Similarly to the proof of Fact 11, an easy calculation yields an upper bound on the sum $\sum_{i=4}^{\infty} 2^{-\alpha_i}$:

$$\sum_{i=4}^{\infty} 2^{-\alpha_i} = 2^{-21} + 2^{-38} + \ldots \leq \sum_{i=1}^{\infty} 2^{-20-i} = 2^{-20}.$$

It follows that

$$677^{1/16} \exp\left(\sum_{i=4}^{\infty} 2^{-\alpha_i}\right) \leq 677^{1/16} \exp\left(2^{-20}\right) \leq 1.50284,$$

as required. □

Fact 13. *The following holds:*

$$\max_{2 \leq n \leq 16} P(n)^{\frac{1}{n-1}} = P(9)^{1/8} = 26^{1/8} = 1.50269\ldots$$

Proof. Using the values of $P(n)$ from recurrence (1), we verify the following inequalities:

$$\begin{aligned}
P(2) &= 1 \text{ and } P(2)^{1/1} = 1 \leq 26^{1/8} = 1.50269\ldots \\
P(3) &= 2 \text{ and } P(3)^{1/2} = 2^{1/2} = 1.4142\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(4) &= 3 \text{ and } P(4)^{1/3} = 3^{1/3} = 1.4422\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(5) &= 5 \text{ and } P(5)^{1/4} = 5^{1/4} = 1.4953\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(6) &= 7 \text{ and } P(6)^{1/5} = 7^{1/5} = 1.4757\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(7) &= 11 \text{ and } P(7)^{1/6} = 11^{1/6} = 1.4913\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(8) &= 16 \text{ and } P(8)^{1/7} = 16^{1/7} = 1.4859\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(9) &= 26 \text{ and } P(9)^{1/8} = 26^{1/8} = 1.50269\ldots \\
P(10) &= 36 \text{ and } P(10)^{1/9} = 36^{1/9} = 1.4890\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(11) &= 56 \text{ and } P(11)^{1/10} = 56^{1/10} = 1.4956\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(12) &= 81 \text{ and } P(12)^{1/11} = 81^{1/11} = 1.4910\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(13) &= 131 \text{ and } P(13)^{1/12} = 131^{1/12} = 1.5012\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(14) &= 183 \text{ and } P(14)^{1/13} = 183^{1/13} = 1.4929\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(15) &= 287 \text{ and } P(15)^{1/14} = 287^{1/14} = 1.4981\ldots \leq 26^{1/8} = 1.50269\ldots \\
P(16) &= 417 \text{ and } P(16)^{1/15} = 417^{1/15} = 1.4951\ldots \leq 26^{1/8} = 1.50269\ldots
\end{aligned}$$

□

Fact 14. *The following holds:*

$$\max_{17 \leq n \leq 32} P(n)^{\frac{1}{n-1}} = P(17)^{1/16} = 677^{1/16} = 1.50283\ldots$$

Proof. Using the values of $P(n)$ from recurrence (1), we verify the following inequalities:

$$
\begin{aligned}
&P(17) = 677 \text{ and } P(17)^{1/16} = 677^{1/16} = 1.50283\ldots\\
&P(18) = 937 \text{ and } P(18)^{1/17} = 937^{1/17} = 1.4955\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(19) = 1457 \text{ and } P(19)^{1/18} = 1457^{1/18} = 1.4988\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(20) = 2107 \text{ and } P(20)^{1/19} = 2107^{1/19} = 1.4959\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(21) = 3407 \text{ and } P(21)^{1/20} = 3407^{1/20} = 1.5018\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(22) = 4759 \text{ and } P(22)^{1/21} = 4759^{1/21} = 1.4966\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(23) = 7463 \text{ and } P(23)^{1/22} = 7463^{1/22} = 1.4998\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(24) = 10843 \text{ and } P(24)^{1/23} = 10843^{1/23} = 1.4977\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(25) = 17603 \text{ and } P(25)^{1/24} = 17603^{1/24} = 1.5027\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(26) = 24373 \text{ and } P(26)^{1/25} = 24373^{1/25} = 1.4978\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(27) = 37913 \text{ and } P(27)^{1/26} = 37913^{1/26} = 1.5000\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(28) = 54838 \text{ and } P(28)^{1/27} = 54838^{1/27} = 1.4980\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(29) = 88688 \text{ and } P(29)^{1/28} = 88688^{1/28} = 1.5021\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(30) = 123892 \text{ and } P(30)^{1/29} = 123892^{1/29} = 1.4983\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(31) = 194300 \text{ and } P(31)^{1/30} = 194300^{1/30} = 1.5006\ldots \le 677^{1/16} = 1.50283\ldots\\
&P(32) = 282310 \text{ and } P(32)^{1/31} = 282310^{1/31} = 1.4990\ldots \le 677^{1/16} = 1.50283\ldots
\end{aligned}
$$

□

Fact 15. *For $2 \le n \le 16$, we have*

$$P(n)\, \exp\left((P(n))^{-1}\, 677^{-\frac{33-n}{16}}\right) \le 677^{\frac{n-1}{16}}. \tag{18}$$

Proof. Let

$$x_n = (P(n))^{-1}\, 677^{\frac{n-1}{16}}, \text{ for } 2 \le n \le 16. \tag{19}$$

Then (18) is equivalent to

$$\exp\left(\frac{x_n}{677^2}\right) \le x_n, \text{ for } 2 \le n \le 16. \tag{20}$$

By Fact 13, we have

$$P(n)^{\frac{1}{n-1}} \le P(9)^{1/8} = 26^{1/8}, \text{ for } 2 \le n \le 16.$$

and this implies

$$x_n = (P(n))^{-1}\, 677^{\frac{n-1}{16}} \ge \frac{677^{\frac{n-1}{16}}}{26^{\frac{n-1}{8}}} = \left(\frac{677}{676}\right)^{\frac{n-1}{16}} \ge \left(\frac{677}{676}\right)^{\frac{1}{16}} = 1.00009\ldots, \text{ for } 2 \le n \le 16.$$

Obviously, we also have $x_n \le 677$, for $n = 2, \ldots, 16$, thus x_n is bounded as follows:

$$\left(\frac{677}{676}\right)^{\frac{1}{16}} \le x_n \le 677, \text{ for } 2 \le n \le 16.$$

To verify (20), we distinguish two cases:

Case 1: $x_n \in \left[\left(\frac{677}{676}\right)^{\frac{1}{16}}, 2\right]$. Then

$$\exp\left(\frac{x_n}{677^2}\right) \leq \exp\left(\frac{2}{677^2}\right) = 1.0000043\ldots \leq \left(\frac{677}{676}\right)^{\frac{1}{16}} = 1.00009\ldots \leq x_n,$$

as required by (20).

Case 2: $x_n \in [2, 677]$. Then

$$\exp\left(\frac{x_n}{677^2}\right) \leq \exp\left(\frac{677}{677^2}\right) = \exp\left(\frac{1}{677}\right) = 1.0014\ldots \leq 2 \leq x_n,$$

as required by (20). □