Computational Geometry Column 64

Adrian Dumitrescu*

Abstract

This column is devoted to minimum partitions of colored point sets into monochromatic parts with pairwise disjoint convex hulls. These partitions offer compact color signatures that in principle could be used for classification and retrieval of discretized images.

Keywords: Crossing-free partition, convex partition, colored point set, color signature.

1 Introduction

A partition of a planar point set is called *noncrossing* (or *crossing-free*), if the convex hulls of the individual parts are disjoint. Crossing-free partitions for uncolored points have a long history; see [1, 2, 5, 7, 8, 10, 11]. Even if it is not explicitly stated, the crossing-free property sometimes follows from the constraints imposed on a class of partitions; see for instance [2]. Several crossing-free partitions of a set of 6 points are depicted in Figure 1. We denote by `cfp`(S) the number of crossing-free partitions of S.

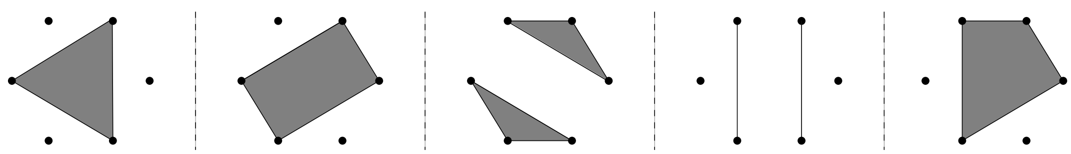

Figure 1: Five crossing-free partitions of a set of 6 points; into 4, 3, 2, 4, and 3 parts, respectively.

Every noncrossing matching (perfect or not) is a crossing-free partition, as well; the parts in such a partition are the segments in the matching and the remaining points as singletons. García, Noy, and Tejel [6] showed that the *double chain* point configuration (made from two convex chains of size $n/2$ with opposite orientations) has $\Theta^*(3^n)$ perfect matchings[1]. Sharir and Welzl [10] showed that the same point configuration has $\Theta^*(4^n)$ matchings. The current best lower and upper bounds on the maximum number of crossing-free partitions, $\Omega(5.23^n)$ and $O(12.24^n)$, are due to the same authors [10]; the lower bound comes from an exhaustive counting of such partitions in the double chain. Many related results concerning various classes of crossing-free geometric subgraphs can be found in [4].

Here we only discuss two-colored point sets, where each point is colored, say, by *red* or *blue*, and crossing-free partitions into monochromatic parts. Among these partitions we focus on *minimum*

*Department of Computer Science, University of Wisconsin–Milwaukee, USA. Email: `dumitres@uwm.edu`

[1]The Θ^*, O^*, Ω^* notation is used to describe the asymptotic growth of functions ignoring polynomial factors.

partitions, i.e., those with a minimum number of parts for the respective point sets. Such a partition is then referred to as a *minimum (noncrossing) partition into monochromatic parts.*

It is known [3] that any two-colored set of n points in the plane, with no three collinear points, can be partitioned into $\lceil \frac{n+1}{2} \rceil$ noncrossing monochromatic parts. Moreover, the $\lceil \frac{n+1}{2} \rceil$ bound is tight. Similarly, any two-colored set of n points in $\mathbb{R}^3$, with no four coplanar points, can be partitioned into $n/3 + O(1)$ monochromatic parts, whose convex hulls are disjoint; the current best lower bound is sublinear [3].

2 Colored point sets, partitions, and signatures

A *color signature* $(\sigma(S), \mu(S))$ may be associated to any finite (two-colored) point set S, where $\sigma(S)$ is the the number of parts in a minimum partition of S into monochromatic parts, and $\mu(S)$ is the number of such partitions. The shortest signature, $(1, 1)$, corresponds to a monochromatic point configuration. See Figures 2 and 3 for some examples. To recognize or classify a given image or point configuration, one would first compute its signature (or an approximation of it) and then compare it against stored signatures that are close to it.

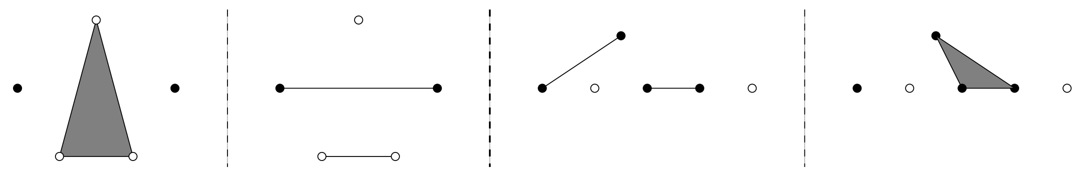

Figure 2: Left: a two-colored set of 5 points with color signature $(3, 2)$; a minimum partition has 3 monochromatic parts, and there are 2 such partitions. Right: a two-colored set of 6 points (with 5 points on a line) whose color signature is $(4, 2)$; a minimum partition has 4 monochromatic parts, and there are 2 such partitions.

Remarks. First, the color signature defined above is invariant under translation, rotation, and scaling. Second, it applies to every point set, e.g., the usual *no three points collinear* restriction is not needed. Third, the same signature scheme applies to multiple colors and is invariant with respect to permutation of the colors. Finally, a standard encoding scheme such as $(\sigma, \mu) \to 2^\sigma 3^\mu$ can be used to obtain integer signatures.

Point sets with many minimum partitions. Consider a set $\tilde{S}$ of $n = 2m$ points placed on a circle, and having alternating colors, red and blue: $\tilde{S} = R \cup B$. We call this configuration *alternating*. Denote by $p(\tilde{S})$ the minimum number of parts necessary to partition $\tilde{S}$. It was shown in [3] than $p(\tilde{S}) \geq m + 1 = n/2 + 1$. We next exhibit an exponential number of minimum partitions of $\tilde{S}$ into $m + 1$ monochromatic parts, and thereby obtain:

Theorem 1. *There exist n-element two-colored point sets that admit $\Omega^*(2^n)$ minimum partitions into monochromatic parts.*

Proof. It is known [5, 8] (see also [1, 9]), that if X is set of n points in convex position, then

$$\texttt{cfp}(X) = C_n = \frac{1}{n+1}\binom{2n}{n} = \Theta(4^n n^{-3/2}),$$

where C_n denotes the nth Catalan number. Hence there are $C_m = \Theta(4^m m^{-3/2}) = \Theta(2^n n^{-3/2})$ crossing-free partitions of the m red points in $\tilde{S}$. We show next that each of these partitions, say Π, can be (uniquely) extended to a minimum partition of $\tilde{S}$ into monochromatic parts. See Figure 3 for an illustration of the argument.

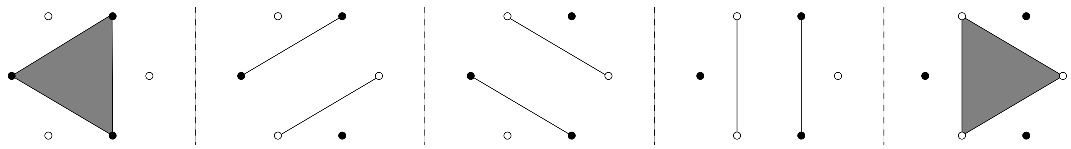

Figure 3: Extending each crossing-free partition of R to a minimum partition of $\tilde{S} = R \cup B$ into monochromatic parts (red points are drawn as filled circles). The color signature of $\tilde{S}$ is $(4, 5)$.

Suppose that Π has k red parts of sizes $x_1, x_2, \ldots, x_k \geq 1$. Since all the n points are in convex position, the k convex hulls of the red parts separate the blue points into $1 + \sum_{i=1}^{k}(x_i - 1)$ subsets with disjoint convex hulls; each subset is made into one part. The resulting partition of $\tilde{S}$ is crossing-free and it consists of k red parts and $1 + \sum_{i=1}^{k} x_i - k = 1 + m - k$ blue parts. Since there are $k + (m + 1 - k) = m + 1$ parts overall, we have obtained a minimum partition of $\tilde{S}$ into monochromatic parts. □

Open problems. As far as we know, the algorithmic complexity of computing the number of crossing-free partitions of an (uncolored) point set is open [10]. We suspect that this status extends to colored point sets and minimum partitions into monochromatic parts.

Since any set of n points in the plane has $O(12.24^n)$ crossing-free partitions [10], this bound also applies to minimum partitions of a two-colored set of n points into monochromatic parts, and thereby gives an initial rough bound for the range of possible color signatures of an n-element two-colored point set.

We summarize these thoughts in a few concluding questions (formulated here for the simplest case of two colors):

1. What is the computational complexity of finding a minimum partition into monochromatic parts, given an n-element two-colored point set? Can a suitable approximation be found efficiently?

2. What is the computational complexity of determining the number of minimum partitions into monochromatic parts, given an n-element two-colored point set? Can a suitable approximation of this number be found efficiently?

3. What is the maximum number of minimum partitions of an n-element two-colored point set into monochromatic parts?

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