Computational Geometry Column 61

Bernardo Ábrego[*] Adrian Dumitrescu[†] Silvia Fernández[*] Csaba D. Tóth[*‡]

Abstract

This column is devoted to the memory of Ferran Hurtado who unexpectedly passed away in 2014. In writing this column we remember some of the problems that he posed or liked. From the many topics Ferran has worked on, we only include a small sample, related to his joint work with the authors of this column.

Keywords: Geometric graph, edge flip, compatible graphs, reconfiguration, colored point set.

1 Introduction

Besides being an accomplished scholar and a generous person, Ferran Hurtado (1951–2014) will be remembered for the beauty and elegance of the many open problems he posed. This column pays homage to him by highlighting some of his favorite problems. Ferran had an exceptional ability to combine simple concepts from geometry and graph theory to create beautiful new problems that required minimal preparation to understand, and yet had broad appeal and often turned out to be quite challenging. The problems combine simple mathematical concepts (such as graphs), simple geometric objects (such as disks or squares), elementary operations (edge flips, edge additions, reconfigurations), and colored point sets.

Many of the problems involve *geometric graphs* $G = (P, E)$, where the vertices are distinct points in the plane, and the edges are straight line segments connecting pairs of points in P. Common geometric graphs include *perfect matchings*, *Hamiltonian cycles*, and *triangulations*; see Fig. 1. A geometric graph is *noncrossing* if edges intersect only at common endpoints (if any). Unless otherwise specified we assume our point sets to be in *general position* in the plane, that is, with no 3 points collinear. A set P is said to be in *convex position* if it is the vertex set of a convex polygon.

Some of the most basic questions concern containment relations between geometric graphs on a point set. Given two families, $\mathcal{A}$ and $\mathcal{B}$, of noncrossing geometric graphs on a fixed point set P, does every graph $G \in \mathcal{A}$ have a subgraph in $\mathcal{B}$, or equivalently, is every graph $H \in \mathcal{B}$ a subgraph of some $G \in \mathcal{A}$? When $n = |P|$ is even, for example, every Hamiltonian cycle on P contains a perfect matching (in fact it contains two); but a noncrossing perfect matching on P need not be part of a noncrossing Hamiltonian cycle (even though every noncrossing matching on $n \geq 4$ vertices is part of a Hamiltonian triangulation in which the Hamiltonian cycle need not use all edges of the

[*]Department of Mathematics, California State University, Northridge, Los Angeles, CA, USA. Email: {bernardo.abrego,silvia.fernandez,csaba.toth}@csun.edu

[†]Department of Computer Science, University of Wisconsin–Milwaukee, USA. Email: dumitres@uwm.edu

[‡]Department of Computer Science, Tufts University, Medford, MA, USA.

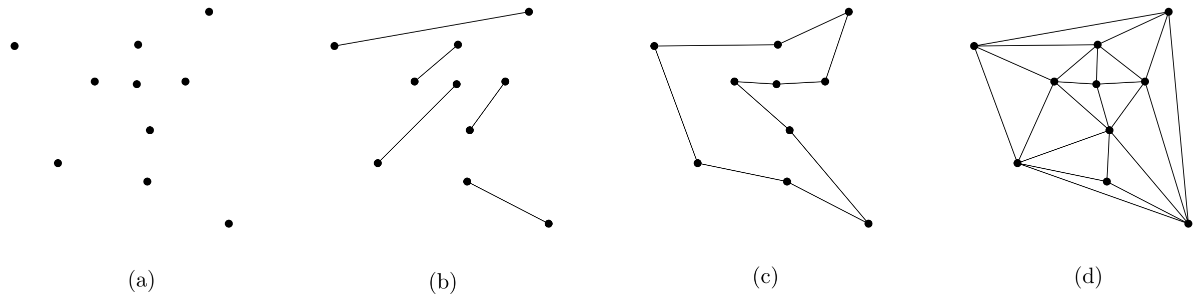


Figure 1: Common geometric graphs on the same point set. (a) A set P of 10 points in the plane in general position; (b) a perfect matching on P; (c) a Hamiltonian cycle on P; (d) Delaunay triangulation of P.

matching [32]). Similarly, every noncrossing Hamiltonian cycle can be augmented to a triangulation, but not all triangulations are Hamiltonian. One can ask for the maximal subgraph in $\mathcal{B}$ contained in every graph in $\mathcal{A}$ (Section 2); or ask how to modify G or H using the minimum number of elementary operations (e.g., edge flips, deletions or additions) to make such a containment possible (Sections 3–4).

2 Matchings in geometric graphs

The complete geometric graph on any n points contains a matching formed by $\Omega(\sqrt{n})$ pairwise crossing segments [15], but it is conjectured that $\Omega(n)$ can be achieved. Every geometric graph with n vertices and no $k+1$ pairwise disjoint edges, $k \leq n/2$, is known to have at most k^2n edges [50]. Let $e_k(n)$ be the smallest number such that every geometric graph with n vertices and $m \geq e_k(n)$ edges contains a noncrossing matching with k edges, where $k \leq n/2$ (Fig. 1, left). It is known that $e_k(n) = \Omega(kn)$ [41] and $e_k(n) = O(k^2n)$ [50], and it is believed that $e_k(n) = \Theta(kn)$.

By exploiting the upper bound $e_k(n) = O(k^2n)$, the edge set of every complete geometric graph on n vertices can be partitioned into $O(n^{3/2})$ noncrossing matchings [14], i.e., each composed of pairwise disjoint segments. It is an exciting open problem to decide whether $O(n)$ such matchings suffice, as in the case of convex complete geometric graphs (where the vertex set is in convex position). It should be noted that even the conjectured bound $e_k(n) = O(kn)$ would only yield a decomposition into $O(n \log n)$ noncrossing matchings [14], and so a new approach is likely needed.

Open problem 1. [15] *Does there exist a constant c such that for any n-element point set, no* 3 *collinear, the complete geometric graph on the n points contains a matching formed by cn pairwise crossing segments?*

Open problem 2. [50] *Does there exist a constant c such that every geometric graph on n vertices and at least ckn edges contains k disjoint edges (i.e., a noncrossing matching with k edges)?*

Open problem 3. [14] *Can the edge set of the complete geometric graph on any n-element point set be partitioned into $O(n)$ noncrossing matchings?*

3 Connectivity augmentation

The objective in typical graph augmentation problems is to add the fewest new edges to a given graph to achieve a desired property [38]. Every noncrossing geometric graph on n vertices in general position can be augmented to a triangulation, which is 2-edge-connected for $n \geq 3$. In general, we cannot hope to increase the vertex-connectivity to 2 or the edge-connectivity to 3, since every triangulation on $n \geq 3$ points in convex position has at least two vertices of degree 2. Abellanas et al. [3] asked what is the minimum number of new edges needed to augment any connected noncrossing geometric graph on at least 3 vertices to a 2-connected or 2-edge-connected one. A tight bound of $n-2$ was shown for 2-connectivity [3], and a tight bound of $\lfloor(2n-2)/3\rfloor$ for 2-edge-connectivity [49]. An edge may obstruct the addition of some other edges, and fewer edges may suffice when we start with an edge-minimal connected graph, i.e., a spanning tree.

Open problem 4. [38] *What is the minimum constant $c > 0$ such that every noncrossing spanning tree on $n \geq 3$ vertices can be augmented to a 2-edge-connected noncrossing geometric graph by adding at most $cn + o(n)$ new edges?*

The current best bounds [38, 49] are $\frac{6}{11} \leq c \leq \frac{2}{3}$. It is not difficult to see that $n-1$ new edges are always sufficient and sometimes necessary to increase the edge-connectivity from 0 to 1. However, the minimum number of edges needed for increasing the edge-connectivity from 0 to 2 is not known.

Open problem 5. [38] *What is the minimum constant $c > 0$ such that every noncrossing geometric graph on $n \geq 3$ vertices can be augmented to a 2-edge-connected noncrossing geometric graph by adding at most $cn + o(n)$ new edges?*

It is known [38] that $c \geq \frac{4}{3}$: a graph that consists of a triangulation on $\lfloor n/3\rfloor$ points and an isolated vertex in each of the $2\lfloor n/3\rfloor - 4$ faces requires $4\lfloor n/3\rfloor - 8$ new edges, two in each face.

4 Edge flips and their relatives

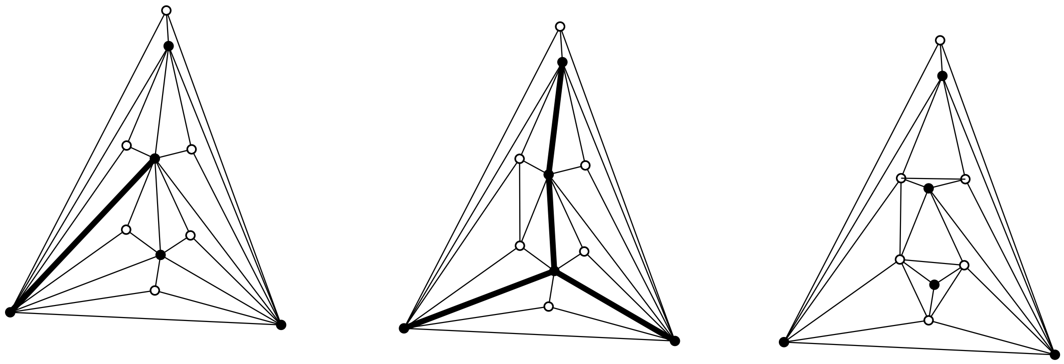
An edge flip in a (noncrossing geometric) triangulation of a point set P is an elementary operation that removes one edge and inserts another edge producing a new triangulation on P [17]; see Fig. 2(a)-(b). In a seminal work, Lawson [42] showed that every triangulation on P can be transformed to any other triangulation on P with $O(n^2)$ edge flips, where $n = |P|$. Hurtado et al. [37] constructed a point set and two triangulations for which $\Omega(n^2)$ flips are necessary. For a set P of points in general position in the plane, the graph of triangulations $\mathcal{T}(P)$ of P has a vertex for every triangulation of P, and two triangulations are adjacent if they differ by a single edge flip. The above results give a tight bound of $\Theta(n^2)$ for the maximum diameter of $\mathcal{T}(P)$ when $|P| = n$.

In the combinatorial setting (where triangulations are edge-maximal planar graphs), the diameter of the graph of triangulations is known to be at most $5.2n$ [19]; this bound crucially relies on the fact that every edge-maximal planar graph can be transformed into a Hamiltonian edge-maximal planar graph with a sequence of at most $n/2$ combinatorial flips [19]. The same approach may be feasible for geometric triangulations, as well.

Open problem 6. [17] *Can every triangulation on n points in the plane be transformed into a Hamiltonian triangulation by a sequence of $o(n^2)$ edge flips?*

Algorithmic questions about reachability in $\mathcal{T}(P)$ for a given point set P are also of interest. For instance, finding the shortest flip sequence is known to be APX-hard [45].

Open problem 7. [17, 45] *Given two triangulations $T_1, T_2 \in \mathcal{T}(P)$, is there a constant-factor approximation algorithm for computing a shortest flip sequence between T_1 and T_2?*



(a) (b) (c)

Figure 2: (a) A triangulation on 11 points that contains no Hamiltonian cycle. (b) Flipping the bold edge in (a) produces this Hamiltonian triangulation. (c) The bold edges in (b) are simultaneously flipped to produce this triangulation.

Simultaneous edge flips. In an edge flip, the removal of an edge merges two triangles, and the resulting convex quadrilateral is retriangulated by a new edge. If two or more edge flips merge disjoint pairs of triangles, then the operations can be performed simultaneously. In every triangulation on n points, at least $(n-4)/5$ edges can be flipped simultaneously [48], and this bound is the best possible [28]. Every triangulation on P can be transformed to any other triangulation on P with $O(n)$ simultaneous flips, where $n = |P|$, and this bound is again, the best possible [28]. It is not known in what measure simultaneous flips can reduce the distance to Hamiltonicity.

Open problem 8. [17] *Can every triangulation on n points in the plane be transformed into a Hamiltonian triangulation by a sequence of $o(n)$ simultaneous edge flips?*

In the combinatorial setting, every edge-maximal planar graph can be transformed into a Hamiltonian edge-maximal graph with a simultaneous flip of $\frac{2}{3}n$ edges [19].

Isomorphic triangulations. The following problem raised by Aichholzer et al. [8] is motivated by possible applications to image analysis and morphing.

Open problem 9. [8] *Is it true that any two planar finite point sets P and Q (in general position) have combinatorially isomorphic triangulations if they have the same size and their convex hulls have the same number of vertices?*

Aichholzer et al. [8] conjectured that the answer is affirmative and managed to prove this in the special case when P and Q have at most 3 interior points. They also constructed, for every $n \geq m \geq 3$, an n-point set $P_{n,m}$ in general position, whose convex hull has m vertices, and with the following property: For every set Q with n points and m vertices on its convex hull, there exist isomorphic triangulations of $P_{n,m}$ and Q. Interestingly, this result does not settle their conjecture because the property of having isomorphic triangulations is not transitive.

Compatible matchings. Two noncrossing geometric graphs, G and H, are *compatible* if their union $G \cup H$ is also noncrossing. For instance, the graphs in Fig. 1(b) and Fig. 1(c) are incompatible, but those in Fig. 1(c) and Fig. 1(d) are compatible.

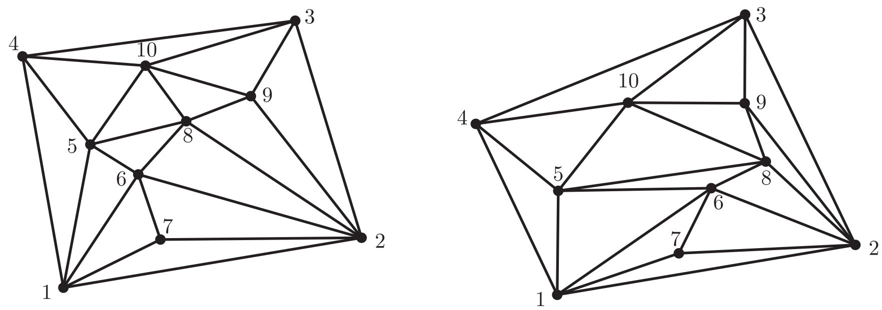


Figure 3: Two sets with combinatorially isomorphic triangulations.

Houle et al. [34] raised the question whether a sequence of some elementary operations can transform a noncrossing perfect matching to any other on the same point set. Given a point set P with an even number of points, let $\mathcal{M} = \mathcal{M}(P)$ be the set of noncrossing perfect matchings on P. A *graph of matchings* on P is $G_E = (\mathcal{M}, E)$ where E contains certain pairs of matchings from $\mathcal{M}$ as specified below. The graph G_E is known to be connected if E consists of pairs of disjoint matchings (that need not be compatible) [34]; or pairs of compatible matchings (that need not be disjoint) [9]. If E consists only of disjoint compatible matchings, then G_E is disconnected when $n \geq 6$ points are in convex position [7], yet it has no isolated vertices when $n \equiv 0 \mod 4$ [39].

If E consists of pairs of matchings whose symmetric difference is a 4-cycle C_4 (i.e., a truly local exchange operation) and P is in convex position, then G_E is again connected [31]. It remains an open problem whether the latter result generalizes to arbitrary point sets in general position.

Open problem 10. [31] *Is the graph $G_E = (\mathcal{M}(P), E)$ connected if E consists of pairs of matchings whose symmetric difference is a C_4?*

5 On the number of triangulations and related questions

Determining or at least estimating the maximum number of noncrossing geometric graphs on n points in the plane is a fundamental question in combinatorial geometry. A classic result of Ajtai et al. [13] shows that the number of noncrossing geometric graphs on n points is $O(c^n)$ for a (large) absolute constant $c > 0$. The constant c has been improved several times since then. The current best bound, $c < 187.53$, is due to Sharir and Sheffer [46].

The power of the *double chain* configuration, depicted in Fig. 4 (left), in establishing good lower bounds for the number of matchings, triangulations, Hamiltonian cycles and trees was first recognized in [29]. For instance, it was widely believed for some time that the double chain gives asymptotically the highest number of triangulations, namely $\Theta^*(8^n)$.

In 2006, another configuration, the so-called *double zig-zag chain*, was shown [11] to admit $\Theta^*(\sqrt{72}^n) = \Omega(8.48^n)$ triangulations[1]. The double zig-zag chain consists of two flat copies of a zig-zag chain; see Fig. 4 (right). A zig-zag chain is the simplest example of an *almost convex* polygon. Such polygons have been introduced and first studied by Hurtado and Noy [36]. By further exploiting the power of *almost convex* polygons, the current best lower bound of $\Omega(8.65^n)$ on the maximum number of triangulations was established in [24]. A similar construction holds

[1]The Θ^*, O^*, Ω^* notation is used to describe the asymptotic growth of functions ignoring polynomial factors.

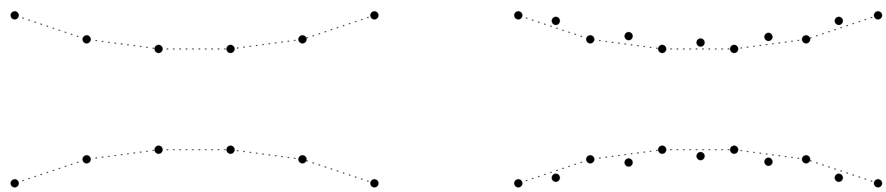


Figure 4: A double chain (left) and a double zig-zag chain (right).

the record for noncrossing perfect matchings [16], and the double chain is still the current best construction for noncrossing Hamiltonian cycles and plane graphs [35].

Open problem 11. [47] *Is the number of* Hamiltonian cycles *smaller than the number of* triangulations *of P, for every point set P?*

Open problem 12. [47] *Does there exist a constant $c < 1$ such that, for every point set P, the number of* perfect matchings *determined by P is at most $O(c^n)$ times the number of* triangulations *of P?*

6 Geometric proximity graphs

Given a finite point set P in the plane, a proximity graph on P is a geometric graph with vertex set P where two points are adjacent if they satisfy a specific proximity condition. Examples of such graphs are the *Delaunay graph*, where two points are adjacent if they are incident to a circle whose interior does not contain any point of P; and the *(modified) Gabriel graph* (MGG), where two points p and q are adjacent if the circle whose diameter is pq does not contains any point of P in its interior.

It is known [18] that the chromatic number of MGG is at most 8 for every finite point set, while the chromatic number of the MGG for sections of the integer lattice is 4. The fact that the clique number of the MGG is at most 4 for every point set, which is attained for sections of the integer lattice, suggests that the chromatic number of an MGG could be strictly greater than 4.

Open problem 13. [18] *Is there a set P whose modified Gabriel graph has chromatic number greater than 4?*

Higher order proximity graphs. Higher order proximity graphs can be defined by relaxing the emptiness condition imposed to some proximity graphs. The *k-Delaunay graph*, k-DG(P), and the *k-Gabriel graph*, k-GG(P), allow the interior of the corresponding circles to contain at most k points of P. Different notions of proximity can be used. In the *k-nearest neighbor graph*, k-NNG(P), each point is adjacent to its k nearest neighbors.

Open problem 14. [18] *What is the smallest integer k such that the k-Gabriel graph of every finite point set is Hamiltonian?*

Dillencourt [21] and Abellanas et al. [2] showed this value to be between 1 and 15. Recently, Kaiser et al. [40] announced that the smallest integer k is between 2 and 10.

Proximity graphs are noncrossing geometric graphs, and higher order proximity graphs are believed to have few crossings. It is known that 1-DG(P) has at least $n - 4$ crossings for every set P of n points in general position and this bound is tight [6].

Open problem 15. *For a constant $k \geq 2$, what is the minimum number of crossings $f_k(n)$ that the k-Delaunay graph of a set of n points in general position can have?*

When n points are in convex position, 1-$\mathsf{DG}(P)$ is known to have $6n - 3\lfloor \frac{n}{2} \rfloor - 19$ crossings and this bound is also tight [6].

Open problem 16. *For a constant $k \geq 2$, what is the minimum number of crossings $f_k(n)$ that the k-Delaunay graph of a set of n points in convex position can have?*

For $k \leq 5$ and $n \geq 44$ there are n-point sets P with planar k-$\mathsf{NNG}(P)$. However, for every n-point set P, the graphs 7-$\mathsf{NNG}(P)$ and 8-$\mathsf{NNG}(P)$ have at least $\frac{n}{2} + \Theta(1)$ and $n + \Theta(1)$ crossings, respectively, and these bounds are tight [6].

Open problem 17. *If $k = 9$ or 10, what is the least number of crossings in the k-nearest neighbor graph of an n-element point set?*

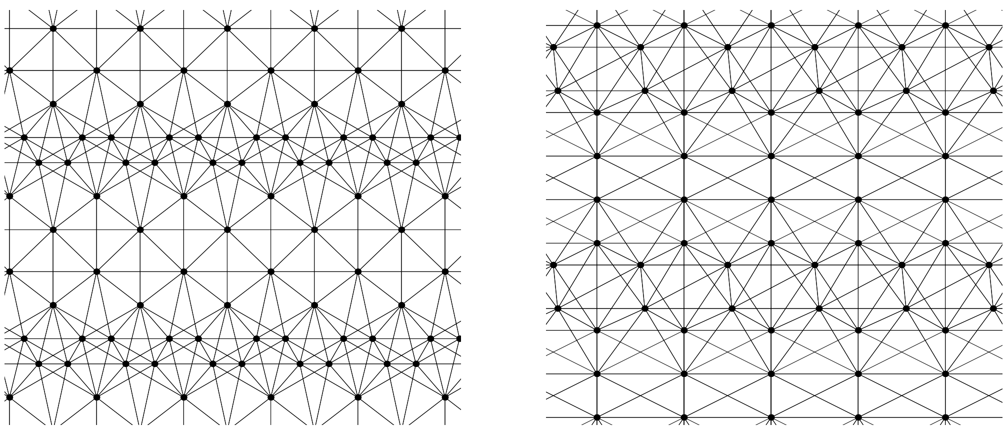


Figure 5: Left: point set whose 9-NNG has $9n/4 + o(n)$ crossings. Right: point set whose 10-NNG has $11n/3 + o(n)$ crossings.

For every n-point set P, the graphs 9-$\mathsf{NNG}(P)$ and 10-$\mathsf{NNG}(P)$ are known to have at least $\frac{13}{6}n + \Theta(1)$ and $\frac{10}{3}n + \Theta(1)$ crossings, respectively, but it is not known whether these bounds can be attained [6]. The current best constructions have $\frac{9}{4}n + \Theta(1)$ and $\frac{11}{3}n + \Theta(1)$ crossings, respectively.

For $k \geq 7$, all higher order proximity graphs mentioned above have $\Omega(k^3 n)$ crossings due to their edge density and the Crossing Lemma. This bound is the best possible up to constant factors, since the n-vertex graphs attaining the Crossing Lemma bound, constructed by Pach and Tóth [43], can be realized as any of these higher order proximity graphs on suitable point sets.

7 Matching points with objects

Given a (usually infinite) family of geometric objects C and a finite set of points P, a *C-matching* of P is a subset $\{C_1, C_2, \ldots, C_k\}$ of C such that each C_i contains exactly two elements of P and each element of P lies in at most one C_i. If each element of P belongs to a C_i, the matching is called *perfect*, and if the C_i's are pairwise disjoint, the matching is called *strong*; see Fig. 6. It is known that any point set of even cardinality admits a perfect C-matching when C is the family of circles or the family of isothetic squares (with respect to a fixed orientation) [4, 5].

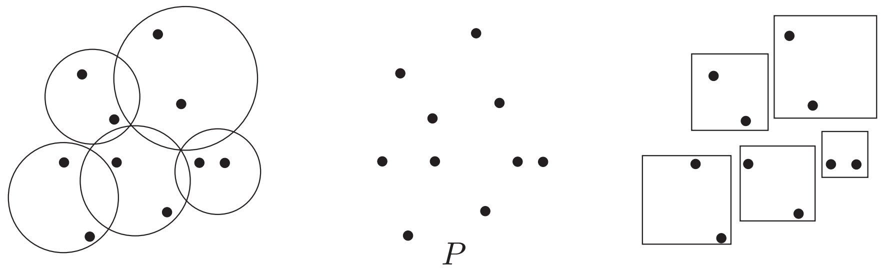


Figure 6: A set P (center) with a perfect circle matching (left) and a strong perfect matching (right).

Open problem 18. [5] *What other classes of convex objects C have the property that any set of points of even cardinality admits a perfect C-matching?*

Strong perfect C-matchings are not always possible. When C is the family of circles, it is known that any set P of n points in general position admits a strong C-matching that uses at least $\frac{1}{4}n + \Theta(1)$ points in P. On the other hand, there exist n-element point sets where at most $\frac{72}{73}n$ points can be strongly matched [4].

Open problem 19. [4] *What is the largest constant c such that every set P of n points in general position admits a strong matching with circles using at least $cn - o(n)$ points of P?*

Thus for circles, the corresponding constant is between $\frac{1}{4}$ and $\frac{72}{73}$.

Open problem 20. [5] *What is the largest constant c such that every set P of n points in general position admits a strong matching with isothetic squares using at least $cn - o(n)$ points of P?*

For isothetic squares, the corresponding constant is between $\frac{2}{5}$ and $\frac{10}{11}$ [5].

8 Convexity and colored point sets

According to the classic result of Erdős and Szekeres [26], every set of at least $\binom{2n-4}{n-2} + 1$ points (in general position) in the plane contains n in convex position. If $f(n)$ denotes the minimum number with this property, it is known [26, 27, 51] that

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-5}{n-2} + 1.$$

In 1975, Erdős [25] asked whether the following sharpening of the Erdős-Szekeres theorem holds: Is there for each $n \geq 3$ a smallest number $g(n)$, such that any set of at least $g(n)$ points in general position in the plane contains an empty convex n-gon? Horton [33] gave a negative answer by constructing arbitrarily large point sets containing no empty convex heptagon; thus $g(n)$ exists only for $n \leq 6$ (the case $n = 6$ was an unresolved mystery for a long time).

Some colored variants of the Erdős-Szekeres problem were considered in [20]. For example, since any 10 points in general position contain an empty convex pentagon [30], any 2-colored 10 points contain a monochromatic empty triangle (as a subset of the above empty pentagon), and so any n points in general position contain $\Omega(n)$ monochromatic empty triangles. It is not an easy matter to see that the number of such triangles must be superlinear in n. This was proved in [10], where a lower bound of $\Omega(n^{5/4})$ was established; the lower bound was subsequently raised to $\Omega(n^{4/3})$ [44].

No similar results hold for 3-colorings, since there exist arbitrarily large 3-colored point sets in general position in the plane with no monochromatic empty triangle [20].

Open problem 21. [10] *Does every two-colored set (say, with $n \geq 10$ points) contain $\Omega(n^2)$ monochromatic empty triangles?*

The question whether every 2-colored set of sufficiently many points contains an empty monochromatic convex quadrilateral remains as an exciting open problem. As a first step, it has been shown that any two-colored set of at least 5044 points determines an empty monochromatic (not necessarily convex) quadrilateral [12].

Open problem 22. [20] *Does every two-colored set of sufficiently many points contain a monochromatic convex empty quadrilateral?*

9 Moving coins

Suppose there are n nonoverlapping coins (of the same denomination, thus indistinguishable) on an infinite table. Any of the coins can be moved by translating it in the plane to a new position, provided no other coins need to be moved out of the way (i.e., no other disk intersects the moving one in its interior). How many moves are needed to bring the n coins from a given start configuration to another given target configuration (also consisting of nonoverlapping coins)? The question was first posed in [1].

From one direction, one can use a so-called *universal algorithm* that moves all coins out to "infinity" and then moves them "back" to occupy the given targets in a suitable order. At most $2n$ translation moves are needed to execute this plan [1], and this is the current best upper bound. See also [22] for other reconfiguration problems.

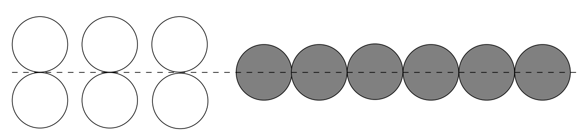


Figure 7: A configuration of 6 congruent disks that requires 9 translation moves.

If the start and target configurations are pairwise disjoint (i.e., no start disk coincides with a target disk), no coin can stay put in its original place, so sometimes n moves are needed. It is also easy to see that the configuration shown in Fig. 7 (the first lower bound that was proposed when the question was asked) requires $3n/2$ moves, since from each pair of tangent disks, the first move must be a nontarget move. An improved lower bound, $\lfloor 8n/5 \rfloor$, appears in [1]; the current best lower bound, $\lfloor 5n/3 \rfloor - 1$, is from [23]. One may find it remarkable that such an easy-to-state problem appears to be out of reach.

Open problem 23. [1, 23] *How many translation moves are needed for the case of n congruent disks? Can the gap between the constant factors $5/3$ and 2 be reduced?*

Instead of disks one can consider the same problem for congruent squares. Here both the start and target configuration consist of n axis-aligned squares. A lower bound of $3n/2$ and an upper bound of $2n$ are readily available [23].

Open problem 24. [22, 23] *How many translation moves are needed for the case of n axis-aligned congruent squares?*

References

[1] M. Abellanas, S. Bereg, F. Hurtado, A. G. Olaverri, D. Rappaport, and J. Tejel, Moving coins, *Computational Geometry: Theory & Applications* **34** (2006), 35–48.

[2] M. Abellanas, P. Bose, A. García, F. Hurtado, C. M. Nicolás, and P. A. Ramos, On structural and graph theoretic properties of higher order Delaunay graphs, *International Journal of Computational Geometry & Applications* **19(06)** (2009), 595–615.

[3] M. Abellanas, A. García, F. Hurtado, J. Tejel, and J. Urrutia, Augmenting the connectivity of geometric graphs, *Computational Geometry: Theory & Applications* **40(3)** (2008), 220–230.

[4] B. M. Ábrego, E. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. Mitchell, and J. Urrutia, Matching points with circles and squares, in *Discrete and Computational Geometry*, selected papers from the Japanese Conference, JCDCG 2004, LNCS 3742, Springer, 2005, pp. 1–15.

[5] B. M. Ábrego, E. Arkin, S. Fernández-Merchant, F. Hurtado, M. Kano, J. Mitchell, and J. Urrutia, Matching points with squares, *Discrete & Computational Geometry* **41** (2009), 77–95.

[6] B. M. Ábrego, R. Fabila-Monroy, S. Fernández-Merchant, D. Flores-Peñaloza, F. Hurtado, V. Sacristán, and M. Saumell, On crossing numbers of geometric proximity graphs, *Computational Geometry: Theory & Applications* **44** (2011), 216–233.

[7] O. Aichholzer, A. Asinowski, and T. Miltzow, Disjoint compatibility graph of non-crossing matchings of points in convex position, *The Electronic Journal of Combinatorics* **22(1)** (2015), #P1.65

[8] O. Aichholzer, F. Aurenhammer, F. Hurtado, and H. Krasser, Towards compatible triangulations, *Theoretical Computer Science* **296(1)** (2003), 3–13.

[9] O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer, F. Hurtado, M. Kano, A. Márquez, D. Rappaport, S. Smorodinsky, D. Souvaine, J. Urrutia, and D.R. Wood, Compatible geometric matchings, *Computational Geometry: Theory & Applications* **42** (2009), 617–626.

[10] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, and J. Urrutia, Empty monochromatic triangles, *Computational Geometry: Theory & Applications* **42** (2009), 934–938.

[11] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber, On the number of plane geometric graphs, *Graphs and Combinatorics* **23(1)** (2007), 67–84.

[12] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber, Large bichromatic point sets admit empty monochromatic 4-gons. *SIAM Journal on Discrete Mathematics* **23(4)** (2010), 2147–2155.

[13] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free subgraphs, *Annals of Discrete Mathematics* **12** (1982), 9–12.

[14] G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy, and J. Urrutia, On the chromatic number of some geometric type Kneser graphs, *Computational Geometry: Theory & Applications* **32** (2005), 59–69.

[15] B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman, Crossing families, *Combinatorica* **14(2)** (1994), 127–134.

[16] A. Asinowski and G. Rote, Point sets with many non-crossing matchings, preprint, February 2015, `arXiv:1502.04925`.

[17] P. Bose and F. Hurtado, Flips in planar graphs, *Computational Geometry: Theory & Applications* **42(1)** (2009), 60–80.

[18] P. Bose, V. Dujmović, F. Hurtado, J. Iacono, S. Langerman, H. Meijer, V. Sacristán, M. Saumell, and D.R. Wood, Proximity graphs: E, δ, Δ, χ, and ω, *International Journal of Computational Geometry & Applications* **22(5)** (2012), 439–469.

[19] J. Cardinal, M. Hoffmann, V. Kusters, Cs. D. Tóth, and M. Wettstein, Arc diagrams, flip distances, and Hamiltonian triangulations, in *Proc. 32nd Symposium on Theoretical Aspects of Computer Science*, LiPICS, 2015, Schloss Dagstuhl, pp. 197–210.

[20] O. Devillers, F. Hurtado, G. Károlyi, and C. Seara, Chromatic variants of the Erdős-Szekeres theorem on points in convex position, *Computational Geometry: Theory & Applications* **26(3)** (2003), 193–208.

[21] M. Dillencourt, A non-Hamiltonian, nondegenerate Delaunay triangulation, *Information Processing Letters* **25** (1987), 149–151.

[22] A. Dumitrescu, Mover problems, in *Thirty Essays in Geometric Graph Theory*, J. Pach, editor, Springer, New York, 2012.

[23] A. Dumitrescu and M. Jiang, On reconfiguration of disks in the plane and other related problems, *Computational Geometry: Theory & Applications* **46(3)** (2013), 191–202.

[24] A. Dumitrescu, A. Schulz, A. Sheffer, and Cs. D. Tóth, Bounds on the maximum multiplicity of some common geometric graphs, *SIAM Journal on Discrete Mathematics* **27(2)** (2013), 802–826.

[25] P. Erdős, On some problems in elementary and combinatorial geometry, *Annali di Matematica Pura ed Applicata* **103** (1975), 99–108.

[26] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935) 463–470.

[27] P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Annales Universitatis Scientiarum Budapestinensis* **3-4** (1960-61), 53–62.

[28] J. Galtier, F. Hurtado, M. Noy, S. Pérennes, and J. Urrutia, Simultaneous edge flipping in triangulations, *International Journal of Computational Geometry & Applications* **13(2)** (2003), 113–133.

[29] A. García, M. Noy and A. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N, *Computational Geometry: Theory & Applications* **16(4)** (2000), 211–221.

[30] H. Harboth, Konvexe Fünfecke in ebenen Punktmengen, *Elemente der Mathematik* **33** (1978), 116–118.

[31] M. C. Hernando, F. Hurtado, and M. Noy, Graphs of non-crossing perfect matchings, *Graphs and Combinatorics* **18(3)** (2002), 517–532.

[32] M. Hoffmann and Cs. D. Tóth, Segment endpoint visibility graphs are Hamiltonian, *Computational Geometry: Theory & Applications* **26(1)** (2003), 47–68.

[33] J. Horton, Sets with no empty convex 7-gons, *Canadian Mathematical Bulletin* **26** (1983), 482–484.

[34] M. E. Houle, F. Hurtado, M. Noy, and E. Rivera-Campo, Graphs of triangulations and perfect matchings, *Graphs and Combinatorics* **21(3)** (2005), 325–331.

[35] C. Huemer and A. de Mier, Lower bounds on the maximum number of non-crossing acyclic graphs, preprint, October 2013, `arXiv:1310.5882`.

[36] F. Hurtado and M. Noy, Counting triangulations of almost-convex polygons, *Ars Combinatoria* **45** (1997), 169–179.

[37] F. Hurtado, M. Noy, and J. Urrutia, Flipping edges in triangulations, *Discrete & Computational Geometry* **22(3)** (1999), 333–346.

[38] F. Hurtado and Cs. D. Tóth, Plane geometric graph augmentation: a generic perspective, in *Thirty Essays on Geometric Graph Theory (J. Pach, ed.)*, Springer, 2013, pp. 327–354.

[39] M. Ishaque, D. L. Souvaine, and Cs. D. Tóth, Disjoint compatible geometric matchings, *Discrete & Computational Geometry* **49(1)** (2013), 89–131.

[40] T. Kaiser, M. Saumell, and N. Van Cleemput, 10-Gabriel graphs are Hamiltonian, preprint, February 2015, `arXiv:1410.0309v3`.

[41] Y. Kupitz, Extremal problems in combinatorial geometry, *Aarhus University Lecture Notes Series*, **53** (1979), Aarhus University, Denmark.

[42] C. Lawson, Transforming triangulations, *Discrete Mathematics* **3** (1972), 365–372.

[43] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17(3)** (1997), 427–439.

[44] J. Pach and G. Tóth, Monochromatic empty triangles in two-colored point sets, *Discrete Applied Mathematics* **161(9)** (2013), 1259–1261.

[45] A. Pilz, Flip distance between triangulations of a planar point set is APX-hard, *Computational Geometry: Theory & Applications* **47(5)** (2014), 589–604.

[46] M. Sharir and A. Sheffer, Counting plane graphs: cross-graph charging schemes, *Combinatorics, Probability & Computing* **22(6)** (2013), 935–954.

[47] M. Sharir, A. Sheffer, and E. Welzl, Counting plane graphs: perfect matchings, spanning cycles, and Kasteleyn's technique, *Journal of Combinatorial Theory, Series A* **102(4)** (2013) 777–794.

[48] D. L. Souvaine, Cs. D. Tóth, and A. Winslow, Simultaneously flippable edges in triangulations, in *Computational Geometry (Hurtado Festschrift)*, LNCS 7579, Springer, 2012, pp. 138–145.

[49] Cs. D. Tóth, Connectivity augmentation in planar straight line graphs, *European Journal of Combinatorics* **33(3)** (2012), 408–425.

[50] G. Tóth, Note on geometric graphs, *Journal of Combinatorial Theory, Series A* **89** (2000), 126-132.

[51] G. Tóth and P. Valtr, The Erdős-Szekeres theorem: upper bounds and generalizations, in *Discrete and Computational Geometry*, J. E. Goodman et al., editors, Cambridge University Press, *MSRI Publications* **52** (2005), 557–568.