Computational Geometry Column 59

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Abstract

This column is about patrolling problems in a geometric network. Mobile agents patrol a road network, and have to visit every point in the network as frequently as possible. The goal is finding a schedule that minimizes the *idle time*, i.e., the maximum time between two consecutive visits of some agent over all points in the network.

Keywords: fence patrolling, idle time, mobile agents, runners, approximation algorithm.

1 Preliminaries

Suppose that a fence (or more generally a road network) needs to be patrolled perpetually by k mobile agents $a_1, \ldots, a_k$, with corresponding maximum speeds $v_1, \ldots, v_k > 0$, so that no point on the fence is left unattended for more than a given amount of time. The problem is to determine if this requirement can be met, and if so, to find a suitable patrolling schedule for the agents. Alternatively, given k mobile agents with maximum speeds $v_1, \ldots, v_k > 0$, one would like to find a schedule that minimizes the *idle time*, i.e., the longest time interval during which some point on the fence is not visited by any agent.

The movement of the agents over the time interval $[0, \infty)$ is described by a *patrolling schedule*, where the speed of the ith agent may vary between zero and its maximum value v_i. Given a network represented by a plane graph G, and a patrolling schedule for k agents, the *idle time* I is the longest time interval in $[0, \infty)$ during which a point on the fence remains unvisited, taken over all points. If ℓ is the total length of the edges in G, and the maximum speeds of the agents are $v_1, \ldots, v_k > 0$, then a straightforward volume argument [9] yields the lower bound $I \geq \ell / \sum_{i=1}^{k} v_i$.

A patrolling schedule with a guaranteed upper bound on the idle time for a finite time interval $[0, t]$ does not necessarily imply the same guarantee over the time interval $[0, \infty)$. To ensure such a guarantee, one usually finds a *periodic* patrolling schedule (as defined subsequently) that can be repeated forever.

Related problems. Multi-agent patrolling is a variation of the problem of multi-robot coverage [4, 5], studied extensively in the robotics community. A variety of models has been considered for patrolling, including deterministic and randomized, as well as centralized and distributed, under various objectives [1, 6, 11]. *Idleness*, as a measure of efficiency for a patrolling strategy, was introduced by Machado et al. [13] in a graph setting; see also the article by Chevaleyre [4].

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The problem of patrolling a circle is reminiscent of the classical *lonely runner conjecture*, introduced by Wills [14] and Cusick [7], independently. Assume that k agents $a_1, \ldots, a_k$, run clockwise along a circle of length 1, starting from the same point at time $t = 0$. They have distinct but constant speeds (the speeds cannot vary, unlike in the model considered earlier). A runner is called *lonely* when he/she is at distance of at least $\frac{1}{k}$ from any other runner (along the circle). The conjecture asserts that each runner a_i is lonely at some time $t_i \in (0, \infty)$. The conjecture has only been confirmed for up to $k = 7$ runners [2, 3, 8].

Fence patrolling. The simplest variant has been considered by Czyzowicz et al. [9], where the road network is a rectifiable Jordan curve, representing a *fence*. The fence is either a *closed* curve (e.g., the boundary of a compact simply connected region in the plane), or an *open* curve (e.g., the boundary between two regions). It can be assumed without loss of generality that the open curve is a line segment and the closed curve is a circle.

We parametrize a line segment and a circle of length ℓ by the interval $[0, \ell]$ (for a closed fence, the endpoints of the interval $[0, \ell]$ are identified). A *unit* circle is a circle of unit length.

A *schedule* for k agents consists of k functions $f_i : [0, \infty) \to [0, \ell]$, for $i = 1, \ldots, k$, where $f_i(t)$ is the position of agent i at time t. Each function f_i is continuous and piecewise differentiable, and its derivative (speed) is bounded by $|f_i'| \leq v_i$. A schedule is called *periodic* with *period* $T > 0$ if $f_i(t) = f_i(t + T)$ for all $i = 1, \ldots, k$ and $t \geq 0$. The *idle time* I of a schedule is the maximum length of an open time interval (t_1, t_2) such that there is a point $x \in [0, \ell]$ where $f_i(t) \neq x$ for all $i = 1, \ldots, k$ and $t \in (t_1, t_2)$.

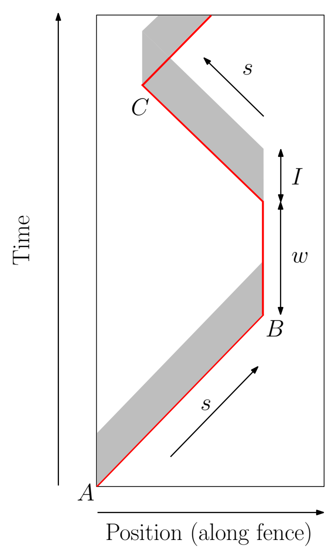

Figure 1: Agent moving with speed s from A to B, waiting at B for time w and then moving from B to C with speed s.

One can use *position-time diagrams* to plot agent trajectories with respect to time. One axis represents the positions $f_i(t)$, $i = 1, \ldots, k$ of the agents along the fence and the other axis represents time. In Fig. 1, for instance, the horizontal axis represents the positions of the agents along the fence and the vertical axis represents time. A schedule with idle time (at most) I is equivalent to a covering problem in such a diagram (see Fig. 1). For a straight-line (i.e., constant speed) trajectory between points (x_1, y_1) and (x_2, y_2) in the diagram, construct a shaded parallelogram with vertices, (x_1, y_1), $(x_1, y_1 + I)$, (x_2, y_2), $(x_2, y_2 + I)$, where I denotes the desired idle time and the shaded region represents the covered region. In particular, if an agent stays put in a time-interval, the parallelogram degenerates to a vertical segment. A schedule for the agents ensures idle time (at most) I if and only if the entire area of the diagram in the time interval $[I, \infty)$ is covered.

2 Patrolling algorithms

Finding a good patrolling schedule is a difficult task that one would like to see automated. Given a fence (open or closed) of length ℓ and maximum speeds $v_1, \ldots, v_k > 0$, a *patrolling algorithm* computes a *patrolling schedule* for the k agents.

Consider an open fence (line segment) of length ℓ and k agents with maximum speeds $v_1, \ldots, v_k > 0$. Czyzowicz et al. [9] proposed a simple partitioning strategy, algorithm $\mathcal{A}_1$, where each agent moves back and forth perpetually in a segment whose length is proportional with its speed. Specifically, algorithm $\mathcal{A}_1$ partitions the segment into k pieces of lengths $\ell v_i / \sum_{j=1}^{k} v_j$ $(i = 1, \ldots, k)$, and schedules the ith agent to patrol the ith interval with speed v_i, starting at an arbitrary point in that interval.

Algorithm $\mathcal{A}_1$ has been proved to be optimal for uniform speeds [9], i.e., when all maximum speeds are equal. Algorithm $\mathcal{A}_1$ achieves an idle time $2\ell / \sum_{i=1}^{k} v_i$ on a segment of length ℓ, and so by the lower bound mentioned earlier, $\mathcal{A}_1$ yields a 2-approximation for the shortest idle time. It has been initially conjectured [9, Conjecture 1] that $\mathcal{A}_1$ is optimal for arbitrary speeds, however this was disproved by Kawamura and Kobayashi [12]: they selected speeds $v_1, \ldots, v_6$ and constructed a schedule for 6 agents that achieves an idle time of $\frac{41}{42}\left(2\ell / \sum_{i=1}^{k} v_i\right)$. This bound was further improved by Dumitrescu, Ghosh and Tóth [10]: for every $\varepsilon > 0$, there exists a patrolling schedule for k agents with suitable speeds $v_1, \ldots, v_k$, that achieves an idle time at most $\left(\frac{24}{25} + \varepsilon\right)\left(2\ell / \sum_{i=1}^{k} v_i\right)$. Moreover, Kawamura and Kobayashi [12] have shown examples with only two distinct speeds for the agents, in which $\mathcal{A}_1$ is suboptimal.

Similarly, no optimal algorithm is known for closed fence patrolling with arbitrary speeds. However, with uniform speeds (i.e., $v_1 = \ldots = v_k = v$), placing the agents uniformly around the circle and letting them move in the same direction yields the shortest idle time. Indeed, the idle time in this case is $\ell/(kv) = \ell / \sum_{i=1}^{k} v_i$, matching the lower bound mentioned earlier.

For the unidirectional variant in which all agents are *required* to move in the same direction (say clockwise) along a circle, Czyzowicz et al. [9, Conjecture 2] conjectured that the following algorithm, $\mathcal{A}_2$, always yields an optimal schedule. Consider without loss of generality a unit circle. Label the agents so that $v_1 \geq v_2 \geq \ldots \geq v_k > 0$. Let r, $1 \leq r \leq k$, be an index such that $\max_{1 \leq i \leq k} i v_i = r v_r$. Algorithm $\mathcal{A}_2$ places the agents $a_1, \ldots, a_r$ at equal distances of $\frac{1}{r}$ around the circle, so that each moves clockwise at the same speed v_r, and discards the remaining agents, if any. Since all agents move in the same direction, we refer to $\mathcal{A}_2$ as the "runners" algorithm. $\mathcal{A}_2$ achieves an idle time of $1 / \max_{1 \leq i \leq k} i v_i$ [9, Theorem 2].

The conjectured optimality of Algorithm $\mathcal{A}_2$ was recently disproved by Dumitrescu, Ghosh and Tóth [10]. Specifically, they constructed a schedule for 32 agents with harmonic speeds $v_i = 1/i$, $i = 1, \ldots, 32$, with an idle time strictly less than 1 (i.e., at most $1 - \varepsilon$, for some small $\varepsilon > 0$). In contrast, since for this setting we have $1 / \max_{1 \leq i \leq k} i v_i = 1 / \max_{1 \leq i \leq k} i \cdot \frac{1}{i} = 1$, Algorithm $\mathcal{A}_2$ yields unit idle time for harmonic speeds, hence it is suboptimal.

As suggested earlier, it seems significantly harder to guarantee a certain idle time in perpetual patrolling than over a finite time interval. For instance, with harmonic speeds, one can achieve any prescribed idle time below 1 for an arbitrarily long time (however, not forever), provided the number of agents k is chosen large enough. To be precise, the following holds [10]. Consider the unit circle, where all agents are required to move in the same direction. For every $0 < \tau \leq 1$ and $t \geq \tau$, there exists $k = k(t) \leq e^{4t/\tau^2}$ and a schedule for the system of k agents with maximum speeds $v_i = \frac{1}{i}$, $i = 1, \ldots, k$, that ensures an idle time at most τ over the time interval $[0, t]$.

3 Open problems

We conclude with a few open problems:

1. **Open (resp., closed) fence patrolling.** Given a (closed or open) fence of length ℓ and k agents with maximum speeds $v_1, \ldots, v_k > 0$, is there a polynomial-time algorithm for computing an optimal patrolling schedule? Is there a polynomial-time approximation scheme? Is there a closed formula for the optimal idle time?

2. **Open fence patrolling.** Observing that all known examples (in [12]) of open fence patrolling outperform the partition-based strategy (i.e., algorithm $\mathcal{A}_1$) only slightly, Kawamura and Kobayashi [12] proposed this revised conjecture. There exists an absolute constant $c > 1$, such that the following holds: the idle time for patrolling an open fence of length ℓ by k agents with maximum speeds $v_1, \ldots, v_k > 0$ is at least $c\,\ell / \sum_{i=1}^{k} v_i$.

3. **Closed fence unidirectional patrolling.** Observing that all known examples (in [10]) of closed fence unidirectional patrolling outperform the runners algorithm $\mathcal{A}_2$ only slightly, we venture the following revised conjecture. The idle time in algorithm $\mathcal{A}_2$ is a constant-factor approximation of the optimal idle time for arbitrary maximum speeds. Further, one can ask: is there a polynomial-time approximation scheme for unidirectional patrolling of the circle?

4. **Discrete cycle patrolling.** Given n stations on the unit circle, each with a prescribed maximum idle time, and k mobile agents, each with a given maximum speed, is there a schedule for the agents such that each station remains unattended for at most its prescribed maximum idle time? Observe that the problem is non-trivial only if $k < n$. More generally, the n stations can be intervals on the unit circle, as in [6].

5. **Graph patrolling.** Given a planar graph embedding (with edge lengths), and k agents with maximum speeds $v_1, \ldots, v_k > 0$, is there a polynomial-time algorithm for computing an optimal patrolling schedule? Is there a polynomial-time approximation scheme? Is there a closed formula for the optimal idle time?

6. **The lonely runner conjecture.** Assume that k runners $a_1, \ldots, a_k$, with distinct but constant speeds, run clockwise along a circle of length 1, starting from the same point at time $t = 0$. Will each runner, a_i, be lonely at some time $t_i \in (0, \infty)$?

7. **Runners in the shade.** In the spirit of the lonely runner conjecture, we pose the following question. Assume that k runners $a_1, \ldots, a_k$, with distinct but constant speeds, run clockwise along a circle of length 1, starting from arbitrary points. Assume also that a certain half of

the circular track (or a certain fixed interval of it) is in the shade at all times. Does there exist a time when all runners are in the shade along the track?

References

[1] N. Agmon, S. Kraus, and G. A. Kaminka, Multi-robot perimeter patrol in adversarial settings, in *Proc. International Conference on Robotics and Automation (ICRA 2008)*, IEEE, 2008, pp. 2339–2345.

[2] J. Barajas and O. Serra, The lonely runner with seven runners, *Electronic Journal of Combinatorics* **15(1)** (2008), R48.

[3] T. Bohman, R. Holzman, and D. Kleitman, Six lonely runners, *Electronic Journal of Combinatorics* **8** (2001), R3.

[4] Y. Chevaleyre, Theoretical analysis of the multi-agent patrolling problem, in *Proc. International Conference on Intelligent Agent Technology (IAT 2004)*, IEEE, 2004, pp. 302–308.

[5] H. Choset, Coverage for robotics—a survey of recent results, *Annals of Mathematics and Artificial Intelligence* **31** (2001), 113–126.

[6] A. Collins, J. Czyzowicz, L. Gasieniec, A. Kosowski, E. Kranakis, D. Krizanc, R. Martin and O. M. Ponce, Optimal patrolling of fragmented boundaries, in *Proc. 25th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA 2013)*, ACM, 2013, pp. 241–250.

[7] T. W. Cusick, View-obstruction problems, *Aequationes Math.* **9** (1973), 165–170.

[8] T. W. Cusick and C. Pomerance, View-obstruction problems III, *Journal of Number Theory* **19** (1984), 131–139.

[9] J. Czyzowicz, L. Gasieniec, A. Kosowski, and E. Kranakis, Boundary patrolling by mobile agents with distinct maximal speeds, in *Proc. 19th European Symposium on Algorithms (ESA 2011)*, LNCS 6942, Springer, 2011, pp. 701–712.

[10] A. Dumitrescu, A. Ghosh, and Cs. D. Tóth, On fence patrolling by mobile agents, preprint, January 2014; `arXiv:1401.6070`.

[11] Y. Elmaliach, N. Agmon, and G. A. Kaminka, Multi-robot area patrol under frequency constraints, in *Proc. International Conference on Robotics and Automation (ICRA 2007)*, IEEE, 2007, pp. 385–390.

[12] A. Kawamura and Y. Kobayashi, Fence patrolling by mobile agents with distinct speeds, in *Proc. 23rd International Symposium on Algorithms and Computation (ISAAC 2012)*, LNCS 7676, Springer, 2012, pp. 598–608.

[13] A. Machado, G. Ramalho, J. D. Zucker, and A. Drogoul, Multi-agent patrolling: an empirical analysis of alternative architectures, *3rd International Workshop on Multi-Agent-Based Simulation*, Springer, 2002, pp. 155–170.

[14] J. M. Wills, Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen, *Monatshefte für Mathematik* **71(3)** (1967), 263–269.