A note on blocking visibility between points

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Abstract

Given a finite point set P in the plane, let $b(P)$ be the smallest number of points $q_1, q_2, \ldots$ not belonging to P which together block all visibilities between elements of P, that is, every open segment whose endpoints belong to P contains at least one point q_i. Let $b(n)$ denote the minimum of $b(P)$ over all n-element point sets P, with no three points on the same line. It is known that $2n - 3 \leq b(n) \leq n2^{c\sqrt{\log n}}$, where c is an absolute constant. Here we raise the lower bound to $\left(\frac{25}{8} - o(1)\right) n$. A better upper bound is obtained for blocking all edges in simple complete topological graphs.

1 Introduction

Let P be a set of n points in the plane, no three of which are collinear. We want to find a small point-set Q, disjoint from P, which blocks all visibilities between pairs of points in P. In other words, every open segment whose endpoints belong to P must contain at least one element of Q. Let $b(P)$ denote the smallest size of such a "blocking" set Q, and let $b(n)$ be the minimum of $b(P)$, over all n-element point sets P, with no three collinear points. Recently, there has been renewed interest in the subject; see, e.g., [5]. However, it is still not known whether $b(n)$ is superlinear in n.

Since each segment connecting a fixed element of P to the other elements must contain a distinct blocking point, we have $b(n) \geq n - 1$. Moreover, all edges of a triangulation of P must be blocked by distinct points. Since every triangulation has at least $2n-3$ edges, it follows that $b(n) \geq 2n-3$. According to Matoušek [5], no better lower bound was known for $b(n)$.

On the other hand, we trivially have $b(n) \leq \binom{n}{2}$. For a finite point set P in the plane, let $\mu(P)$ be the size of the set of *midpoints* of all $\binom{n}{2}$ segments determined by P. Let $\mu(n)$ stand for the minimum of $\mu(P)$, over all n-element point sets P, with no three points collinear. According to a result of Pach [6], $\mu(n) \leq n2^{c\sqrt{\log n}}$, where c is an absolute constant. In other words, for any n, there exists a set of n points in the plane, with no three points collinear, whose set of midpoints is bounded by the above function. This shows that, if $\mu(n)$ is not $O(n)$, it can be only slightly superlinear. Obviously, for any P, the set of midpoints of all segments determined by P blocks all visibilities between point pairs of P, so that

$$b(n) \leq \mu(n) \leq n2^{c\sqrt{\log n}},$$

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where c is an absolute constant.

For points in *convex position*, that is, for the vertex set P of a convex polygon, it is known that $b(P) = \Omega(n \log n)$; see [5]. Indeed, assigning weight $1/i$ to each point pair separated by $i-1$ other vertices of P, it is easy to check that the total weight of all point pairs blocked by a single point is at most 1. Therefore, we have

$$b(P) \geq n \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{i} \geq \frac{n \log n}{2}. \tag{1}$$

This argument that goes back to [4] was implicit in [1], and has been rediscovered by A. Holmsen, R. Pinchasi, G. Tardos, and others. For the *regular* n-gon P_n, it is known [7] that $b(P_n) = \Omega(n^2)$. It is perfectly possible that the same is true for all convex n-gons.

Throughout this note, we always assume that our point sets are in *general position*, that is, no three points are collinear. In Section 2, we raise the lower bound on $b(n)$ from $2n-3$ to $\left(\frac{25}{8} - o(1)\right) n$.

Theorem 1. *$b(n) \geq \left(\frac{25}{8} - o(1)\right) n$.*

A *geometric* graph is a graph drawn by straight-line edges on a set of vertices in the plane in general position. If the edges of G are drawn by continuous arcs connecting the corresponding pair of vertices but not passing through any third vertex, then G is called a *topological* graph. A topological graph is said to be *simple* if any pair of its edges meet at most once, which may be a common endpoint or a common interior point at which the two edges properly cross, but not both. Tangencies between the edges of G are not allowed. If it leads to no confusion the topological graph G and its underlying abstract graph will be denoted by the same letter; see [2, p. 396].

In Section 3, we discuss what happens under a natural relaxation of straight-line visibility. Suppose that we want to block all edges of a simple *complete* topological graph on n vertices in the plane. Is it possible that for some of these graphs $O(n)$ blocking points suffice? More precisely, let $\tilde{b}(n)$ denote the smallest number $\tilde{b}$ for which there exists a simple complete topological graph G on n vertices, and a set of $\tilde{b}$ points different from the vertices of G such that every edge of G passes through at least one of these points.

As in the geometric case, we trivially have $\tilde{b}(n) \geq n-1$. Perhaps $\tilde{b}(n) \geq 2n-3$ also holds. From the other direction, we prove the following.

Theorem 2. *$\tilde{b}(n) = O(n \log n)$.*

2 Proof of Theorem 1

We can assume that $n \geq 10$. Recall that if P' is a set of n points in (strictly) convex position, then $b(P') \geq |P'| \log |P'|/2$. Consider an n-element point set P, and let $P' = \text{conv}(P)$, and $h = |P'|$ be the number of vertices on the convex hull of P. Since every edge of a fixed triangulation must contain at least one blocking point, we have

$$b(P) \geq 3n - h - 3. \tag{2}$$

We distinguish two cases depending on whether h is large or respectively, small, with respect to n. Assume first that $h \geq \frac{25}{2} \frac{n}{\log n}$. Note that $\log h \geq (\log n)/2$. Obviously, $b(P) \geq b(P')$, and the lower bound for the convex case yields:

$$b(P) \geq b(P') \geq \frac{1}{2} \cdot \frac{25}{2} \cdot \frac{n}{\log n} \cdot \frac{\log n}{2} = \frac{25}{8} n,$$

as required. Therefore, we can assume for the rest of the proof that $h \leq \frac{25}{2}\frac{n}{\log n}$. Under this assumption, (2) already gives a better lower bound: $b(P) \geq 3n - h - 3 = 3n - o(n)$.

To further improve this bound, we select a suitable triangulation Δ of the point set, and argue that in addition to the blocking points required by the edges of Δ, a constant fraction of n further blocking points are required. Assume for simplicity that $n = 8k + 2$, for some positive integer k. Pick a point $p_0 \in \text{conv}(P)$, and label the remaining $n-1$ points in clockwise order of visibility from p, as $p_1, p_2, \ldots, p_{n-1}$.

Define k 10-element subsets of P as follows. Let

$$P_i := \{p_0, p_{8i-7}, p_{8i-6}, \ldots, p_{8i+1}\}, \qquad i = 1, 2, \ldots, k.$$

Note that any two consecutive groups, P_i and P_{i+1} share two points.

Consider any group P_i. By an old result of Harboth [3], there exists a 5-element subset $Q_i \subset P_i$ which spans (the vertex set of) an empty convex pentagon $\text{conv}(Q_i)$. For each i, take the 5 edges of $\text{conv}(Q_i)$, and extend the set of these $5k$ edges to a triangulation Δ of P. Since no *three* diagonals of $\text{conv}(Q_i)$ are concurrent, blocking the 5 diagonals of $\text{conv}(Q_i)$ requires (at least) 3 blocking points. That is, in addition to the two points blocking the two edges of Δ inside $\text{conv}(Q_i)$, an extra blocking point is needed for each $i = 1, \ldots, k$. Since the interiors of the k pentagons $\text{conv}(Q_i)$ are pairwise disjoint, it follows that the number of extra blocking points, in addition to the $3n - h - 3$ points required by the edges of the triangulation Δ is at least $k = \lfloor n/8 \rfloor$. Overall, P requires at least $3n - h - 3 + k = \left(\frac{25}{8} - o(1)\right) n$ blocking points, as claimed.

3 Proof of Theorem 2

We recursively construct a sequence of simple complete topological graphs G_i, $i = 0, 1, \ldots$, with the following properties:

(1) G_i has 2^i vertices.

(2) The vertices of G_i have x-coordinates $0, 1, \ldots 2^i - 1$, respectively.

(3) The edges of G_i are drawn as x- and y-monotone curves.

(4) There is a set of at most $i2^i$ points that block all edges of G_i.

Let G_0 be a topological graph with one vertex at $(0, 0)$ and no edges. Suppose that we have already constructed G_i, and we are about to construct G_{i+1}. Apply an affine transformation on G_i such that the x-coordinates of the vertices are $0, 2, 4, \ldots 2^{i+1} - 2$, while the y-coordinates are all very close to 0. Take two copies of this drawing, one translated by $(0, 1)$ and one by $(1, -1)$. The union is a simple but not complete topological graph with 2^{i+1} vertices. The edges are drawn as x- and y-monotone curves. Let $u_0, \ldots u_{2^i-1}$ (resp. $v_0, \ldots v_{2^i-1}$) be the vertices of the upper (resp. lower) copy from left to right. Connect each vertex in the upper copy with each vertex in the lower copy by a straight line segment. Now we have a complete simple topological graph. We "bend" the new edges a little bit so that they can be blocked by few points. Observe that for any j, k, $0 \leq j, k \leq 2^i - 1$, the segment $u_j v_k$ passes very close to the point $(j + k + 1/2, 0)$. For every j, k, $0 \leq j, k \leq 2^i - 1$, substitute the segment $u_j v_k$ by the 2-edge polygonal path $u_j, (j + k + 1/2, 0), v_k$. Let G_{i+1} be the resulting complete topological graph. It is easy to see that the drawing is simple, we have 2^{i+1} vertices with x-coordinates $0, 1, \ldots 2^{i+1} - 1$, and the edges are x- and y-monotone curves. By induction, we know that the edges in the upper (resp. lower) copy can be blocked by $i2^i$ points, and that the points $(m + 1/2, 0)$, $m = 0, \ldots, 2^{i+1} - 2$ block all edges between the two parts. Therefore, $i2^i + i2^i + 2^{i+1} - 1 < (i + 1)2^{i+1}$ points block all edges of G_{i+1}.

This concludes the proof when the number of vertices n is a power of 2. For other values of n, take G_i where $2^{i-1} < n \leq 2^i$, and remove $2^i - n$ vertices.

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