

ANCHORED RECTANGLE AND SQUARE PACKINGS^{*†}

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Abstract

For points p_1, \dots, p_n in the unit square $[0, 1]^2$, an *anchored rectangle packing* consists of interior-disjoint axis-aligned empty rectangles $r_1, \dots, r_n \subseteq [0, 1]^2$ such that point p_i is a corner of the rectangle r_i (that is, r_i is *anchored* at p_i) for $i = 1, \dots, n$. We show that for every set of n points in $[0, 1]^2$, there is an anchored rectangle packing of area at least $7/12 - O(1/n)$, and for every $n \in \mathbb{N}$, there are point sets for which the area of every anchored rectangle packing is at most $2/3$. The maximum area of an anchored *square* packing is always at least $5/32$ and sometimes at most $7/27$.

The above constructive lower bounds immediately yield constant-factor approximations, of $7/12 - \varepsilon$ for rectangles and $5/32$ for squares, for computing anchored packings of maximum area in $O(n \log n)$ time. We prove that a simple greedy strategy achieves a $9/47$ -approximation for anchored square packings, and $1/3$ for lower-left anchored square packings. Reductions to maximum weight independent set (MWIS) yield a QPTAS for anchored rectangle packings in $\exp(\text{poly}(\varepsilon^{-1} \log n))$ time and a PTAS for anchored square packings in $n^{O(1/\varepsilon)}$ time.

Keywords: Rectangle packing, anchored rectangle, greedy algorithm, charging scheme, approximation algorithm.

1 Introduction

Let $P = \{p_1, \dots, p_n\}$ be a finite set of points in an axis-aligned bounding rectangle U . An *anchored rectangle packing* for P is a set of axis-aligned empty rectangles r_1, \dots, r_n that lie in U , are interior-disjoint, and p_i is one of the four corners of r_i for $i = 1, \dots, n$ (a rectangle is *empty* if it does not contain any point from P in its interior); rectangle r_i is said to be *anchored* at p_i .

For a given point set $P \subset U$, we wish to find the maximum total area $A(P)$ of an anchored rectangle packing of P . Since the ratio between areas is an affine invariant, we may assume that $U = [0, 1]^2$. However, if we are interested in the maximum area of an *anchored square packing*, we *must* assume that $U = [0, 1]^2$ (or that the aspect ratio of U is bounded from below by a constant; otherwise, with an *arbitrary* rectangle U , the guaranteed area is only zero).

Finding the maximum area of an anchored rectangle packing of n given points is suspected but not known to be NP-hard. Balas and Tóth [10] showed that the number of distinct rectangle

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packings that attain the maximum area, $A(P)$, can be exponential in n . From the opposite direction, the same authors [10] proved an exponential upper bound on the number of maximum area configurations, namely $2^n C_n = \Theta(8^n/n^{3/2})$, where $C_n = \frac{1}{n+1} \binom{2n}{n} = \Theta(4^n/n^{3/2})$ is the n th Catalan number. Note that a greedy strategy may fail to find $A(P)$; see Fig. 1.

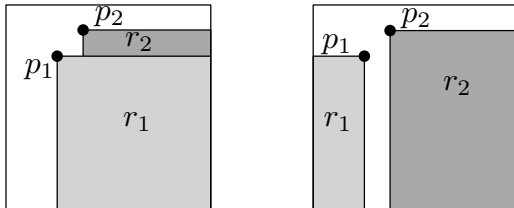


Figure 1: For $P = \{p_1, p_2\}$, with $p_1 = (\frac{1}{4}, \frac{3}{4})$ and $p_2 = (\frac{3}{8}, \frac{7}{8})$, a greedy algorithm selects rectangles of area $\frac{3}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{5}{8} = \frac{41}{64}$ (left), which is less than the area $\frac{1}{4} \cdot \frac{3}{4} + \frac{5}{8} \cdot \frac{7}{8} = \frac{47}{64}$ of the packing on the right.

Variants and generalizations. We consider three additional variants of the problem. An *anchored square packing* is an anchored rectangle packing in which all rectangles are squares; a *lower-left anchored rectangle packing* is a rectangle packing where each point $p_i \in r_i$ is the lower-left corner of r_i ; and a *lower-left anchored square packing* has both properties. We suspect that all variants, with rectangles or with squares, are NP-hard. Here, we put forward several approximation algorithms, while it is understood that the news regarding NP-hardness can occur at any time or perhaps take some time to establish.

The problem can be generalized to other geometric shapes with distinct representatives. Let $P = \{p_1, \dots, p_n\}$ be a finite set of points in a compact domain $U \subset \mathbb{R}^d$, and let $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_n\}$ be n families of measurable sets (e.g., rectangles, squares, or disks) such that for all $r \in \mathcal{F}_i$, we have $p_i \in r \subseteq U$ and $\mu(r) \geq 0$ is the measure of r . An *anchored packing for (P, \mathcal{F})* is a set of pairwise interior-disjoint representatives r_1, \dots, r_n with $r_i \in \mathcal{F}_i$ for $i = 1, \dots, n$. We wish to find an anchored packing for (P, \mathcal{F}) of maximum measure $\sum_{i=1}^n \mu(r_i)$. While some variants are trivial (e.g., when $U = [0, 1]^2$ and \mathcal{F}_i consists of all rectangles containing p_i), there are many interesting and challenging variants (e.g., when \mathcal{F}_i consists of disks containing p_i ; or when U is nonconvex). In this paper we assume that the domain U and the families \mathcal{F} are axis-aligned rectangles or axis-aligned squares in the plane.

Anchored packing with	rectangles	squares	LL-rect.	LL-sq.
Guaranteed max. area	$\frac{7}{12} - O(\frac{1}{n}) \leq A(n) \leq \frac{2}{3}$	$\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$	0	0
Greedy approx. ratio	$7/12 - \varepsilon$	9/47	0.091 [24]	1/3
Approximation scheme	QPTAS	PTAS	QPTAS	PTAS

Table 1: Table of results for the four variants studied in this paper. The last two columns refer to lower-left anchored rectangles and lower-left anchored squares, respectively.

Contributions. Our results are summarized in Table 1.

(i) We first deduce upper and lower bounds on the maximum area of an anchored rectangle packing of n points in $[0, 1]^2$. For $n \in \mathbb{N}$, let $A(n) = \inf_{|P|=n} A(P)$. We prove that $\frac{7}{12} - O(1/n) \leq A(n) \leq \frac{2}{3}$ for all $n \in \mathbb{N}$ (Sections 2 and 3).

(ii) Let $A_{\text{sq}}(P)$ be the maximum area of an anchored square packing for a point set P , and $A_{\text{sq}}(n) = \inf_{|P|=n} A_{\text{sq}}(P)$. We prove that $\frac{5}{32} \leq A_{\text{sq}}(n) \leq \frac{7}{27}$ for all n (Sections 2 and 4).

(iii) The above constructive lower bounds immediately yield constant-factor approximations for computing anchored packings of maximum area ($7/12 - \varepsilon$ for rectangles and $5/32$ for squares) in $O(n \log n)$ time (Sections 3 and 4). In Section 5 we show that a (natural) greedy algorithm for anchored square packings achieves a better approximation ratio, namely $9/47 = 1/5.22\dots$, in $O(n^2)$ time. By refining some of the tools developed for this bound, in Section 6 we prove a tight bound of $1/3$ for the approximation ratio of a greedy algorithm for lower-left anchored square packings.

(iv) We obtain a polynomial-time approximation scheme (PTAS) for the maximum area anchored square packing problem, and a quasi-polynomial-time approximation scheme (QPTAS) for the maximum area anchored rectangle packing problem, via a reduction to the maximum weight independent set (MWIS) problem for axis-aligned squares [17] and rectangles [3, 20], respectively. Given n points, an $(1 - \varepsilon)$ -approximation for the anchored square packing of maximum area can be computed in time $n^{O(1/\varepsilon)}$; and for the anchored rectangle packing of maximum area, in time $\exp(\text{poly}(\varepsilon^{-1} \log n))$. Both results extend to the lower-left anchored variants (Section 7).

Motivation and related work. Packing axis-aligned rectangles in a rectangular container, albeit without anchors, is the unifying theme of several classic optimization problems. The 2D knapsack problem, strip packing, and 2D bin packing involve arranging a set of given rectangles in the most economic fashion [3, 11]. The maximum area independent set (MAIS) problem for rectangles (squares, or disks, etc.), is that of selecting a maximum area packing from a given set [5]; see classic papers such as [8, 35, 36, 37, 38] and also more recent ones [12, 13, 22] for quantitative bounds and constant approximations. These optimization problems are NP-hard, and there is a rich literature on approximation algorithms. Given an axis-parallel rectangle U in the plane containing n points, the problem of computing a maximum-area empty axis-parallel sub-rectangle contained in U is one of the oldest problems studied in computational geometry [7, 19]; the higher dimensional variant has been also studied [23]. In contrast, our problem setup is fundamentally different: the rectangles (one for each anchor) have variable sizes, but their location is constrained by the anchors.

Map labeling problems in geographic information systems (GIS) [31, 32, 34] call for choosing interior-disjoint rectangles that are incident to a given set of points in the plane. GIS applications often impose constraints on the label boxes, such as aspect ratio, minimum and maximum size, or priority weights. Most optimization problems of such variants are known to be NP-hard [26, 29, 30, 33]. In this paper, we focus on maximizing the total area of an anchored rectangle packing.

In a restricted setting where each point p_i is the lower-left corner of the rectangle r_i and $(0,0) \in P$, Allen Freedman [39, 40] conjectured almost 50 years ago that there is a lower-left anchored rectangle packing of area at least $1/2$. The current best lower bound on the area under these conditions is (about) 0.091, as established in [24]. The analogous problem of estimating the total area for lower-left anchored square packings is much easier. If P consists of the n points $(i/n, i/n)$, $i = 0, 1, \dots, n-1$, then the total area of the n anchored squares is at most $1/n$, and so it tends to zero as n tends to infinity. A looser anchor restriction, often appearing in map labeling problems with square labels, requires the anchors to be contained in the boundaries of the squares, however the squares need to be congruent; see e.g., [6, 41].

In the context of *covering* (as opposed to *packing*), the problem of covering a given polygon by disks of given centers and varying radii such that the sum of areas of the disks is minimized has been considered in [1, 15]. In particular, covering $[0, 1]^2$ with ℓ_∞ -disks of given centers and minimal area as in [14] is dual to the anchored square packings studied in this paper.

Notation. Given an n -element point set P contained in $U = [0, 1]^2$, denote by $\text{OPT} = \text{OPT}(P)$ a packing (of rectangles or squares, as the case may be) of maximum total area. An algorithm for a packing problem has approximation ratio α if the packing it computes, Π , satisfies $\text{area}(\Pi) \geq \alpha \text{area}(\text{OPT})$, for some $\alpha \leq 1$. A set of points is *in general position* if no two points have the same x - or y -coordinate. The boundary of a planar body B is denoted by ∂B , and its interior by $\text{int}(B)$.

2 Upper Bounds

Proposition 1. *For every $n \in \mathbb{N}$, there exists a point set P_n such that every anchored rectangle packing for P_n has area at most $\frac{2}{3}$. Consequently, $A(n) \leq \frac{2}{3}$.*

Proof. Consider the point set $P = \{p_1, \dots, p_n\}$, where $p_i = (x_i, y_i) = (2^{-i}, 2^{-i})$, for $i = 1, \dots, n$; see Fig. 2 (left). Let $R = \{r_1, \dots, r_n\}$ be an anchored rectangle packing for P . Since $p_1 = (\frac{1}{2}, \frac{1}{2})$, any rectangle anchored at p_1 has height at most $\frac{1}{2}$, width at most $\frac{1}{2}$, and hence area at most $\frac{1}{4}$.

For $i = 2, \dots, n$, the x -coordinate of p_i , x_i , is halfway between 0 and x_{i-1} , and y_i is halfway between 0 and y_{i-1} . Consequently, if p_i is the lower-right, lower-left or upper-left corner of r_i , then $\text{area}(r_i) \leq (\frac{1}{2^i})(1 - \frac{1}{2^i}) = \frac{1}{2^i} - \frac{1}{4^i}$. If p_i is the upper-right corner of r_i , then $\text{area}(r_i) \leq \frac{1}{4^i}$. Therefore, in all cases, we have $\text{area}(r_i) \leq \frac{1}{2^i} - \frac{1}{4^i}$. The total area of an anchored rectangle packing is bounded from above as follows:

$$A(P) \leq \sum_{i=1}^n \left(\frac{1}{2^i} - \frac{1}{4^i} \right) = \left(1 - \frac{1}{2^n} \right) - \frac{1}{3} \left(1 - \frac{1}{4^n} \right) = \frac{2}{3} - \frac{1}{2^n} + \frac{1}{3 \cdot 4^n} \leq \frac{2}{3}. \quad \square$$

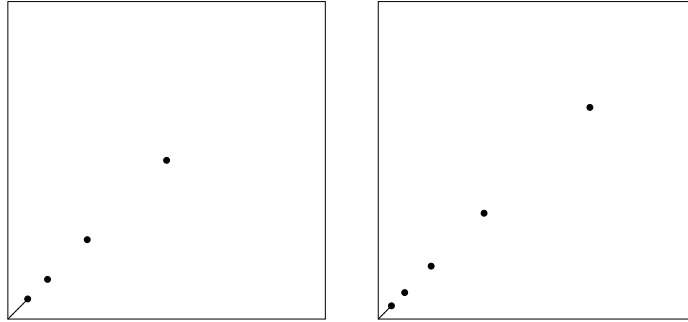


Figure 2: Left: $\frac{2}{3}$ upper bound construction for anchored rectangles. Right: $\frac{7}{27}$ upper bound construction for anchored squares.

Proposition 2. *For every $n \in \mathbb{N}$, there exists a point set P_n such that every anchored square packing for P_n has area at most $\frac{7}{27}$. Consequently, $A_{\text{sq}}(n) \leq \frac{7}{27}$.*

Proof. Consider the point set $P = \{p_1, \dots, p_n\}$, where $p_i = (\frac{4}{3} \cdot 2^{-i}, \frac{4}{3} \cdot 2^{-i})$, for $i = 1, \dots, n$; see Fig. 2 (right). Let $S = \{s_1, \dots, s_n\}$ be an anchored square packing for P . Since $p_1 = (\frac{2}{3}, \frac{2}{3})$ and $p_2 = (\frac{1}{3}, \frac{1}{3})$, any square anchored at p_1 or at p_2 has side-length at most $\frac{1}{3}$, hence area at most $\frac{1}{9}$. For $i = 3, \dots, n$, the x -coordinate of p_i , x_i , is halfway between 0 and x_{i-1} , and y_i is halfway between 0 and y_{i-1} . Hence any square anchored at p_i has side-length at most $x_i = y_i = \frac{4}{3 \cdot 2^i}$, hence area at most $\frac{16}{9 \cdot 4^i}$. The total area of an anchored square packing is bounded from above as follows:

$$A_{\text{sq}}(P) \leq \frac{2}{9} + \frac{1}{9} \sum_{j=1}^{n-1} \frac{1}{4^j} < \frac{2}{9} + \frac{1}{9} \sum_{j=1}^{\infty} \frac{1}{4^j} = \frac{2}{9} + \frac{1}{9} \cdot \frac{1}{3} = \frac{7}{27}. \quad \square$$

Remark. Stronger upper bounds hold for small n , e.g., $n \in \{1, 2\}$. Specifically, $A(1) = A_{\text{sq}}(1) = 1/4$ attained for the center $(\frac{1}{2}, \frac{1}{2}) \in [0, 1]^2$, and $A(2) = 4/9$ and $A_{\text{sq}}(2) = 2/9$ attained for $P = \{(\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})\}$.

3 Lower Bound for Anchored Rectangle Packings

In this section, we prove that for every set P of n points in $[0, 1]^2$, we have $A(P) \geq \frac{7n-2}{12(n+1)}$. Our proof is constructive; we give a divide & conquer algorithm that partitions U into horizontal strips and finds n anchored rectangles of total area bounded from below as required. We start with a weaker lower bound, of about $1/2$, and then sharpen the argument to establish the main result of this section, a lower bound of about $7/12$.

Proposition 3. *For every set of n points in the unit square $[0, 1]^2$, an anchored rectangle packing of area at least $\frac{n}{2(n+1)}$ can be computed in $O(n \log n)$ time.*

Proof. Let $P = \{p_1, \dots, p_n\}$ be a set of points in the unit square $[0, 1]^2$ sorted by their y -coordinates. Draw a horizontal line through each point in P ; see Fig. 3 (left). These lines divide $[0, 1]^2$ into $n+1$ horizontal strips. A strip can have zero width if two points have the same x -coordinate. We leave a narrowest strip empty and assign the remaining strips to the n points such that each rectangle above (resp., below) the chosen narrowest strip is assigned to a point of P on its bottom (resp., top) edge. For each point divide the corresponding strip into two rectangles with a vertical line through the point. Assign the larger of the two rectangles to the point.

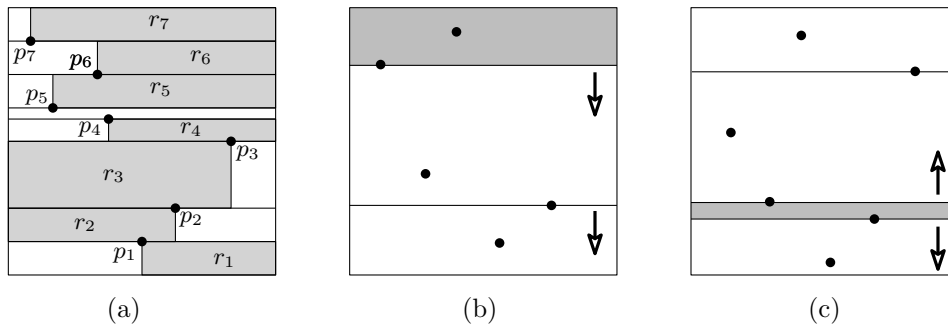


Figure 3: Left: Horizontal strips with anchored rectangles for 7 points. Middle: an example of the partition for odd n (here $n = 5$). Right: an example of the partition for even n (here $n = 6$). The strip that is discarded is shaded in the drawing.

The area of narrowest strip is at most $\frac{1}{n+1}$. The rectangle in each of the remaining n strips covers at least $\frac{1}{2}$ of the strip. This yields a total area of at least $\frac{n}{2(n+1)}$. \square

For a stronger lower bound, our key observation is that for *two* points in a horizontal strip, one can always pack two anchored rectangles in the strip that cover strictly more than half the area of the strip. Specifically, we prove the following easy-looking statement with 2 points in a rectangle (however, we did not find an easy proof!). The proof of Lemma 1 is deferred to Appendix A.

Lemma 1. *Let $P = \{p_1, p_2\}$ be two points in an axis-parallel rectangle R with one of the points on the boundary of R . Then there exist two empty rectangles in R anchored at the two points of total area at least $\frac{7}{12} \text{area}(R)$, and this bound is the best possible.*

In order to partition the unit square into strips that contain two points, one on the boundary, we need to use parity arguments. Let P be a set of n points in $[0, 1]^2$ with y -coordinates $0 \leq y_1 \leq y_2 \leq \dots \leq y_n \leq 1$. Set $y_0 = 0$ and $y_{n+1} = 1$. For $i = 1, \dots, n+1$, put $h_i = y_i - y_{i-1}$, namely h_i is the i th vertical gap. Obviously, we have

$$h_i \geq 0 \text{ for all } i = 1, \dots, n+1, \text{ and } \sum_{i=1}^{n+1} h_i = 1. \quad (1)$$

Parity considerations are handled by the following lemma.

Lemma 2. (i) *If n is odd, then at least one of the following $(n+1)/2$ inequalities is satisfied:*

$$h_i + h_{i+1} \leq \frac{2}{n+1}, \text{ for (odd) } i = 1, 3, \dots, n-2, n. \quad (2)$$

(ii) *If n is even, then at least one of the following $n+2$ inequalities is satisfied:*

$$h_1 \leq \frac{2}{n+2}, \quad h_{n+1} \leq \frac{2}{n+2}, \quad h_i + h_{i+1} \leq \frac{2}{n+2}, \text{ for } i = 1, 2, \dots, n. \quad (3)$$

Proof. Assume first that n is odd. Put $a = \frac{2}{n+1}$ and assume that none of the inequalities in (2) is satisfied. Summation would yield $\sum_{i=1}^{n+1} h_i > \frac{n+1}{2} a = 1$, which is an obvious contradiction.

Assume now that n is even. Put $a = \frac{2}{n+2}$ and assume that none of the inequalities in (3) is satisfied. Summation would yield $2 \sum_{i=1}^{n+1} h_i > (n+2)a = 2$, an obvious contradiction. \square

We can now prove the main result of this section.

Theorem 1. *For every set of n points in the unit square $[0, 1]^2$, an anchored rectangle packing of area at least $\frac{7(n-1)}{12(n+1)}$ when n is odd and $\frac{7n}{12(n+2)}$ when n is even can be computed in $O(n \log n)$ time.*

Proof. Let $P = \{p_1, \dots, p_n\}$ be a set of points in the unit square $[0, 1]^2$ sorted by their y -coordinates with the notation introduced above. By Lemma 2, we find a horizontal strip corresponding to one of the inequalities that is satisfied.

Assume first that n is odd. Draw a horizontal line through each point in $p_j \in P$, for j even, as shown in Fig. 3. These lines divide $[0, 1]^2$ into $\frac{n+1}{2}$ rectangles (horizontal strips). Suppose now that the satisfied inequality is $h_i + h_{i+1} \leq \frac{2}{n+2}$ for some odd i . Then we leave a rectangle between $y = y_{i-1}$ and $y = y_{i+1}$ empty, i.e., r_i is a rectangle of area 0. For the remaining rectangles, we assign two consecutive points of P such that each strip above $y = y_{i+1}$ (resp., below $y = y_{i-1}$) is assigned a point of P on its bottom (resp., top) edge. Within each rectangle R , we can choose two anchored rectangles of total area at least $\frac{7}{12} \text{area}(R)$ by Lemma 1. By Lemma 2(i), the area of the narrowest strip is at most $\frac{2}{n+1}$. Consequently, the area of the anchored rectangles is at least $\frac{7}{12} \left(1 - \frac{2}{n+1}\right) = \frac{7(n-1)}{12(n+1)}$.

Assume now that n is even. If the selected horizontal strip corresponds to the inequality $h_1 \leq \frac{2}{n+2}$, then divide the unit square along the lines $y = y_i$, where i is odd. We leave the strip of height h_1 empty, and assign pairs of points to all remaining strips such that one of the two points lies on the top edge of the strip. We proceed analogously if the inequality $h_{n+1} \leq \frac{2}{n+2}$ is satisfied. Suppose now that the satisfied inequality is $h_i + h_{i+1} \leq \frac{2}{n+2}$. If i is odd, we leave the strip of height $h_i \leq \frac{2}{n+2}$ (between $y = y_{i-1}$ and $y = y_i$) empty; if i is even, we leave the strip of height $h_{i+1} \leq \frac{2}{n+2}$ (between $y = y_i$ and $y = y_{i+1}$) empty. Above and below the empty strip, we can form a total of $n/2$ strips, each containing two points of P , with one of the two points lying on the bottom or the top edge of the strip. By Lemma 2(i), the area of the empty strip is at most $\frac{2}{n+2}$. Consequently, the area of the anchored rectangles is at least $\frac{7}{12} \left(1 - \frac{2}{n+2}\right) = \frac{7n}{12(n+2)}$, as claimed. \square

4 Lower Bound for Anchored Square Packings

Given a set $P \subset U = [0,1]^2$ of n points, we show there is an anchored square packing of large total area. The proof we present is constructive; we give a recursive partitioning algorithm (as an inductive argument) based on a quadtree subdivision of U that finds n anchored squares of total area at least $5/32$. We need the following easy fact:

Observation 1. *Let $u, v \subseteq U$ be two congruent axis-aligned interior-disjoint squares sharing a common edge such that $u \cap P \neq \emptyset$ and $\text{int}(v) \cap P = \emptyset$. Then $u \cup v$ contains an anchored empty square whose area is at least $\text{area}(u)/4$.*

Proof. Let a denote the side-length of u (or v). Assume that v lies right of u ; see Fig. 4. Let

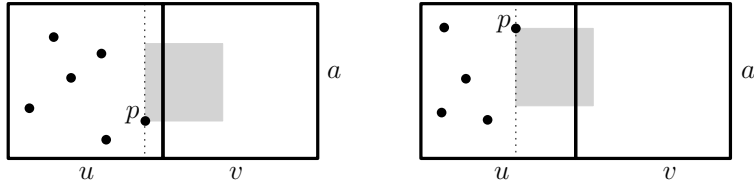


Figure 4: Two adjacent congruent squares u and v , of side length a , and $p \in P$ is the rightmost point in u . Left: p lies in the lower half-rectangle of u . Right: p lies in the upper half-rectangle of u .

$p \in P$ be the rightmost point in u . If p lies in the lower half-rectangle of u then the square of side-length $a/2$ whose lower-left anchor is p is empty and has area $a^2/4$. Similarly, if p lies in the higher half-rectangle of u then the square of side-length $a/2$ whose upper-left anchor is p is empty and has area $a^2/4$. \square

Theorem 2. *For every set of n points in $U = [0,1]^2$, where $n \geq 1$, an anchored square packing of total area at least $5/32$ can be computed in $O(n \log n)$ time.*

Proof. We first derive a lower bound of $1/8$ and then sharpen it to $5/32$. The first few paragraphs of the proof are the same for both lower bounds ($1/8$ and $5/32$).

We proceed by induction on the number of points n contained in U and assigned to U ; during the subdivision process, the role of U is taken by any subdivision square. If all points in P lie on U 's boundary, ∂U , pick one arbitrarily, say, $(x, 0)$; assume $x \leq 1/2$. (All assumptions in the proof are made without loss of generality.) Then the square $[x, x + 1/2] \times [0, 1/2]$ is empty and its area is $1/4$, where $1/4 > 5/32 > 1/8$, as required. Otherwise, discard the points in $P \cap \partial U$ and continue on the remaining points.

If $n = 1$, we can assume that $x(p), y(p) \leq 1/2$. Then the square of side-length $1/2$ whose lower-left anchor is p is empty and contained in U , as desired; hence $A_{\text{sq}}(P) \geq 1/4$. If $n = 2$, let x_1, x_2, x_3 be the widths of the 3 vertical strips determined by the two points, where $x_1 + x_2 + x_3 = 1$. We can assume that $0 \leq x_1 \leq x_2 \leq x_3$; then there are two anchored empty squares with total area at least $x_2^2 + x_3^2 \geq 2/9$, where $2/9 > 5/32 > 1/8$, as required.

Assume now that $n \geq 3$. Subdivide U into four congruent squares, U_1, \dots, U_4 , labeled counterclockwise around the center of U according to the quadrant containing the square. Partition P into four subsets P_1, \dots, P_4 such that $P_i \subset U_i$ for $i = 1, \dots, 4$, with ties broken arbitrarily.

An initial lower bound of $1/8$. We next derive the lower bound $A_{\text{sq}}(P) \geq 1/8$. We distinguish 4 cases, depending on the number of empty sets P_i .

Case 1: precisely one of P_1, \dots, P_4 is empty. We can assume that $P_1 = \emptyset$. By Observation 1, $U_1 \cup U_2$ contains an empty square anchored at a point in $P_1 \cup P_2$ of area at least $\text{area}(U_1)/4 = 1/16$. By induction, U_3 and U_4 each contain an anchored square packing of area at least $c \cdot \text{area}(U_3) = c \cdot \text{area}(U_4)$. Overall, we have $A_{\text{sq}}(P) \geq 2c/4 + 1/16 \geq c$, which holds for $c \geq 1/8$.

Case 2: precisely two of P_1, \dots, P_4 are empty. We can assume that the pairs $\{P_1, P_2\}$ and $\{P_3, P_4\}$ each consist of one empty and one nonempty set. By Observation 1 $U_1 \cup U_2$ and $U_3 \cup U_4$, respectively, contain a square anchored at a point in $P_1 \cup P_2$ and $P_3 \cup P_4$ of area at least $\text{area}(U_0)/4 = 1/16$. Hence $A_{\text{sq}}(P) \geq 2 \cdot \frac{1}{16} = 1/8$.

Case 3: precisely three of P_1, \dots, P_4 are empty. We can assume that $P_3 \neq \emptyset$. Let $(a, b) \in P$ be a maximal point in the product order (e.g., the sum of coordinates is maximum). Then $s = [a, a + \frac{1}{2}] \times [b, b + \frac{1}{2}]$ is a square anchored at (a, b) , $s \subseteq [0, 1]^2$ since $(a, b) \in U_3$, and $\text{int}(s) \cap P = \emptyset$. Hence $A_{\text{sq}}(P) \geq \text{area}(s) = 1/4$.

Case 4: $P_i \neq \emptyset$ for every $i = 1, \dots, 4$. Note that $A_{\text{sq}}(P) \geq \sum_{i=1}^4 A_{\text{sq}}(P_i)$, where the squares anchored at P_i are restricted to U_i . Induction completes the proof in this case.

In all four cases, we have verified that $A_{\text{sq}}(P) \geq 1/8$, as claimed. The inductive proof can be turned into a recursive algorithm based on a quadtree subdivision of the n points, which can be computed in $O(n \log n)$ time [9, 21]. In addition, computing an extreme point (with regard to a specified axis-direction) in any subsquare over all needed such calls can be executed within the same time bound. Note that the bound in Case 3 is at least $5/32$ and Case 4 is inductive.

Sharpening the analysis of Cases 1 and 2 yields an improved bound $5/32$; since $5/32 < 1/4$, the value $5/32$ is not a bottleneck for Cases 3 and 4. Details are given in Appendix B; the running time remains $O(n \log n)$. \square

5 Constant-Factor Approximations for Anchored Square Packings

In this section we investigate better approximations for square packings. Given a finite point set $P \subset [0, 1]^2$, perhaps the most natural greedy strategy for computing an anchored square packing of large area is the following.

Algorithm 1. Set $Q = P$ and $S = \emptyset$. While $Q \neq \emptyset$, repeat the following. For each point $q \in Q$, compute a *candidate* square $s(q)$ such that (i) $s(q) \subseteq [0, 1]^2$ is *anchored* at q , (ii) $s(q)$ is empty of points from P in its interior, (iii) $s(q)$ is interior-disjoint from all squares in S , and (iv) $s(q)$ has maximum area. Then choose a largest candidate square $s(q)$, and a corresponding point q , and set $Q \leftarrow Q \setminus \{q\}$ and $S \leftarrow S \cup \{s(q)\}$. When $Q = \emptyset$, return the set of squares S .

Remark. Let ρ_1 denote the approximation ratio of Algorithm 1, if it exists. The construction in Fig. 5(a–b) shows that $\rho_1 \leq 1/4$. For a small $\varepsilon > 0$, consider the point set $P = \{p_1, \dots, p_n\}$, where $p_1 = (1/2 + \varepsilon, 1/2 + \varepsilon)$, $p_2 = (1/2, 0)$, $p_3 = (0, 1/2)$, and the rest of the points lie on the lower side of U in the vicinity of p_2 , i.e., $x_i \in (1/2 - \varepsilon/2, 1/2 + \varepsilon/2)$ and $y_i = 0$ for $i = 4, \dots, n$. The packing generated by Algorithm 1 consists of a single square of area $(1/2 + \varepsilon)^2$, as shown in Fig. 5(a), while the packing in Fig. 5(b) has an area larger than $1 - \varepsilon$. By letting ε be arbitrarily small, we deduce that $\rho_1 \leq 1/4$.

We first show that Algorithm 1 achieves a ratio of $1/6$ (Theorem 3) using a charging scheme that compares the greedy packing with an optimal packing. We then refine our analysis and sharpen the approximation ratio to $\frac{9}{47} = 1/5.22\dots$ (Theorem 4).

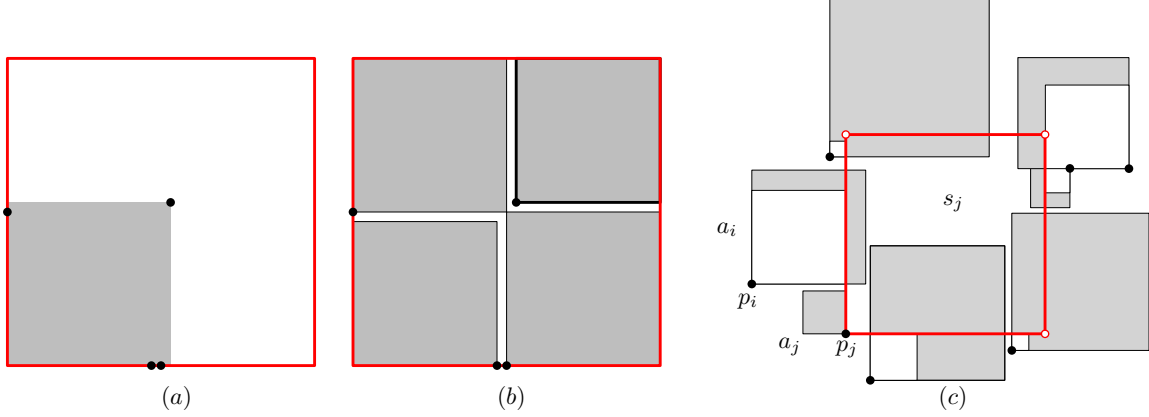


Figure 5: (a–b) A $1/4$ upper bound for the approximation ratio of Algorithm 1. (c) Charging scheme for Algorithm 1. Without loss of generality, the figure illustrates the case when s_j is a lower-left anchored square.

Charging scheme for the analysis of Algorithm 1. Label the points in $P = \{p_1, \dots, p_n\}$ and the squares in $S = \{s_1, \dots, s_n\}$ in the order in which they are processed by Algorithm 1 with $q = p_i$ and $s_i = s(q)$. Let $G = \sum_{i=1}^n \text{area}(s_i)$ be the area of the greedy packing, and let OPT denote an optimal packing with $A = \text{area}(\text{OPT}) = \sum_{i=1}^n \text{area}(a_i)$, where a_i is the square anchored at p_i .

We employ a charging scheme, where we distribute the area of every optimal square a_i with $\text{area}(a_i) > 0$ among some greedy squares; and then show that the total area of the optimal squares charged to each greedy square s_j is at most $6 \text{area}(s_j)$ for all $j = 1, \dots, n$. (Degenerate optimal squares, i.e., those with $\text{area}(a_i) = 0$ do not need to be charged). For each step $j = 1, \dots, n$ of Algorithm 1, we shrink some of the squares a_1, \dots, a_n , and charge the area-decrease to the greedy square s_j . By the end (after the n th step), each of the squares a_1, \dots, a_n will be reduced to a single point.

Specifically in step j , Algorithm 1 chooses a square s_j , and: (1) we shrink square a_j to a single point; and (2) we shrink every square a_i , $i > j$ that intersects s_j in its interior until it no longer does so. This procedure ensures that no square a_i , with $i < j$, intersects s_j in its interior in step j . Refer to Fig. 5(c). Observe three important properties of the above iterative process:

- (i) After step j , the squares $s_1, \dots, s_j, a_1, \dots, a_n$ have pairwise disjoint interiors.
- (ii) After step j , we have $\text{area}(a_j) = 0$ (since a_j was shrunk to a single point).
- (iii) At the beginning of step j , if a_i intersects s_j in its interior (and so $i \geq j$), then $\text{area}(a_i) \leq \text{area}(s_j)$ since s_j is feasible for p_i when a_j is selected by Algorithm 1 due to the greedy choice.

Lemma 3. *Suppose there exists a constant $\varrho \geq 1$ such that for every $j = 1, \dots, n$, square s_j receives a charge of at most $\varrho \text{area}(s_j)$. Then Algorithm 1 computes an anchored square packing whose area G is at least $1/\varrho$ times the optimal.*

Proof. Overall, each square s_j receives a charge of at most $\varrho \text{area}(s_j)$ from the squares in an optimal solution. Consequently, $A = \text{area}(\text{OPT}) = \sum_{i=1}^n \text{area}(a_i) \leq \varrho \sum_{j=1}^n \text{area}(s_j) = \varrho G$, and thus $G \geq A/\varrho$, as claimed. \square

In the remainder of this section, we bound the charge received by one square s_j ($j = 1, \dots, n$). We distinguish two types of squares a_i , $i > j$, whose area is reduced by s_j :

$$\mathcal{Q}_1 = \{a_i : i > j, \text{ the area of } a_i \text{ is reduced by } s_j, \text{ and } \text{int}(a_i) \text{ contains no corner of } s_j\},$$

$\mathcal{Q}_2 = \{a_i : i > j, \text{ the area of } a_i \text{ is reduced by } s_j, \text{ and } \text{int}(a_i) \text{ contains a corner of } s_j\}$.

It is clear that if the insertion of s_j reduces the area of a_i , $i > j$, then a_i is in either \mathcal{Q}_1 or \mathcal{Q}_2 . Note that the area of a_j is also reduced to 0, but it is in neither \mathcal{Q}_1 nor \mathcal{Q}_2 .

Lemma 4. *Each square s_j receives a charge of at most $6 \text{ area}(s_j)$.*

Proof. Consider the squares in \mathcal{Q}_1 . Assume that a_i intersects the interior of s_j , and it is shrunk to a'_i . The area-decrease $a_i \setminus a'_i$ is an L-shaped region, at least half of which lies inside s_j ; see Fig. 5. By property (i), the L-shaped regions are pairwise interior-disjoint; and hence the sum of their areas is at most $2 \text{ area}(s_j)$. Consequently, the area-decrease caused by s_j in squares in \mathcal{Q}_1 is at most $2 \text{ area}(s_j)$.

Consider the squares in \mathcal{Q}_2 . There are at most three squares a_i , $i > j$, that can contain a corner of s_j since the anchor of s_j is not contained in the interior of any square a_i . Since the area of each square in \mathcal{Q}_2 is at most $\text{area}(s_j)$ by property (iii), the area decrease is at most $3 \text{ area}(s_j)$, and so is the charge received by s_j from squares.

Finally, $\text{area}(a_j) \leq \text{area}(s_j)$ by property (iii), and this is also charged to s_j . Overall s_j receives a charge of at most $6 \text{ area}(s_j)$. \square

The combination of Lemmas 3 and 4 readily implies the following.

Theorem 3. *Algorithm 1 computes an anchored square packing whose area is at least $1/6$ times the optimal.*

Refined analysis of the charging scheme. We next improve the upper bound for the charge received by s_j ; we assume for convenience that $s_j = U = [0, 1]^2$. For the analysis, we use only a few properties of the optimal solution. Specifically, assume that a_1, \dots, a_m are interior-disjoint squares such that each a_i : (a) intersects the interior of s_j ; (b) has at least a corner in the exterior of s_j ; (c) does not contain $(0, 0)$ in its interior; and (d) $\text{area}(a_i) \leq \text{area}(s_j)$.

The intersection of any square a_i with ∂U is a polygonal line on the boundary ∂U , consisting of one or two segments. Since the squares a_i form a packing, these intersections are interior-disjoint.

Let $\Delta_1(x)$ denote the maximum area-decrease of a set of squares a_i in \mathcal{Q}_1 , whose intersections with ∂U have total length x . Similarly, let $\Delta_2(x)$ denote the maximum area-decrease of a set of squares a_i in \mathcal{Q}_2 , whose intersections with ∂U have total length x . By adding suitable squares to \mathcal{Q}_1 , we can assume that $4 - x$ is the total length of the intersections $a_i \cap \partial U$ over squares in \mathcal{Q}_2 (i.e., the squares in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ cover the entire boundary of U). Consequently, the maximum total area-decrease is given by

$$\Delta(x) = \Delta_1(x) + \Delta_2(4 - x), \text{ and } \Delta = \sup_{0 \leq x \leq 4} \Delta(x). \quad (4)$$

We now derive upper bounds for $\Delta_1(x)$ and $\Delta_2(x)$ independently, and then combine these bounds to optimize $\Delta(x)$. Since the total perimeter of U is 4, the domain of $\Delta(x)$ is $0 \leq x \leq 4$.

Lemma 5. *The following inequalities hold:*

$$\Delta_1(x) \leq 2, \quad (5)$$

$$\Delta_1(x) \leq x, \quad (6)$$

$$\Delta_1(x) \leq 1 + (x - 1)^2, \text{ for } 1 \leq x \leq 2, \quad (7)$$

$$\Delta_1(x) \leq 1 + \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4}, \text{ for } 0 \leq x \leq 4. \quad (8)$$

Proof. Inequality (5) was explained in the proof of Theorem 3. Inequalities (6) and (7) follow from the fact that the side-length of each square a_i is at most 1 and from the fact that the area-decrease is at most the area (of respective squares); in addition, we use the inequality $\sum x_i^2 \leq (\sum x_i)^2$, for $x_i \geq 0$ and $\sum x_i \leq 1$, and the inequality $x^2 + y^2 \leq 1 + (x + y - 1)^2$, for $0 \leq x, y \leq 1$, and $x + y > 1$.

Write

$$\Delta_1(x) = \Delta_1^{\text{in}}(x) + \Delta_1^{\text{out}}(x), \quad (9)$$

where $\Delta_1^{\text{in}}(x)$ and $\Delta_1^{\text{out}}(x)$ denote the maximum area-decrease contained in U and the complement of U , respectively, of a set of squares in \mathcal{Q}_1 whose intersections with ∂U have total length x , where $0 \leq x \leq 4$. Obviously, $\Delta_1^{\text{in}}(x) \leq \text{area}(U) = 1$. We next show that

$$\Delta_1^{\text{out}}(x) \leq \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4},$$

and thereby establish inequality (8).

Consider a square a_i of side-length $x_i \leq 1$ in \mathcal{Q}_1 . Let z_i denote the length of the shorter side of the rectangle $a_i \setminus U$. The area-decrease outside U equals $x_i z_i - z_i^2$ and so it is bounded from above by $x_i^2/4$ (equality is attained when $z_i = x_i/2$).

Consequently,

$$\Delta_1^{\text{out}}(x) \leq \sup_{\substack{0 \leq x_i \leq 1 \\ \sum x_i = x}} \sum \frac{x_i^2}{4} = \frac{\lfloor x \rfloor}{4} + \frac{(x - \lfloor x \rfloor)^2}{4},$$

where the last equality follows from a standard weight-shifting argument, and equality is attained when x is subdivided into $\lfloor x \rfloor$ unit length intervals and a remaining shorter interval of length $x - \lfloor x \rfloor$. \square

Let $k \leq 3$ be the number of squares a_i in \mathcal{Q}_2 , where $i > j$. We can assume that exactly 3 squares a_i , with $i > j$, are in \mathcal{Q}_2 , one for each corner except the lower-left anchor corner of U , that is, $k = 3$; otherwise the proof of Lemma 4 already yields an approximation ratio of $1/5$. Clearly, we have $\Delta_2(x) \leq k \leq 3$, for any x .

We first bring the squares in \mathcal{Q}_2 into *canonical position*: x monotonically decreases, $\Delta(x)$ does not decrease, and properties (a–d) listed earlier are maintained. Specifically, we transform each square $a_i \in \mathcal{Q}_2$ as follows (refer to Fig. 6):

- Move the anchor of a_i to another corner if necessary so that one of its coordinates is contained in the interval $(0, 1)$;
- translate a_i horizontally or vertically so that $a_i \cap U$ decreases to a skinny rectangle of width ε , for some small $\varepsilon > 0$.

Lemma 6. *The following inequality holds:*

$$\Delta_2(x) \leq 2x - \frac{x^2}{3}, \quad \text{for } 0 \leq x \leq 4. \quad (10)$$

Proof. Assume that the squares in \mathcal{Q}_2 are in canonical position. Let y_i denote the side-length of a_i , let x_i denote the length of the longer side of the rectangle $a_i \cap U$ and z_i denote the length of the shorter side of the rectangle $a_i \setminus U$, $i = 1, 2, 3$. Since the squares in \mathcal{Q}_2 are in canonical position, we have $x_i + z_i = y_i \leq 1$, for $i = 1, 2, 3$. We also have $\sum_{i=1}^3 x_i = x - O(\varepsilon)$. Letting $\varepsilon \rightarrow 0$, we have $\sum_{i=1}^3 x_i = x$.

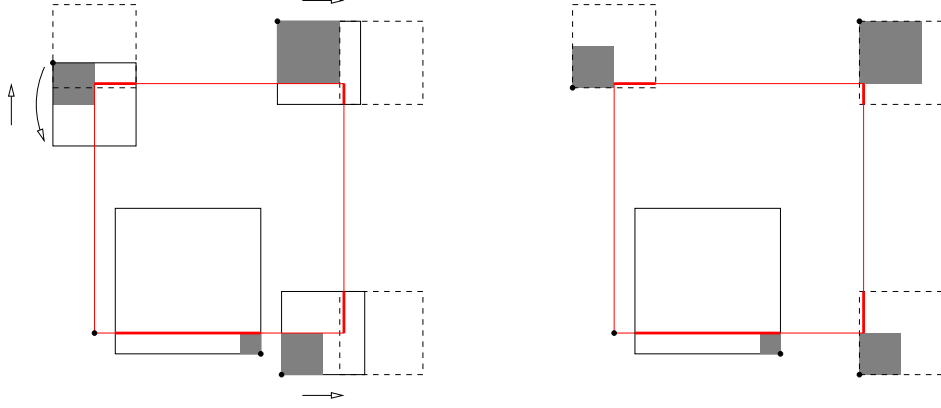


Figure 6: Bounding the area-decrease; moving the squares in \mathcal{Q}_2 into canonical position. The parts of the boundary of U covered by each square (after transformations) are drawn in thick red lines.

$$\begin{aligned}
 \Delta_2(x) &= \sup_{x_i+z_i=y_i \leq 1} \sum_{i=1}^3 (y_i^2 - z_i^2) = \sup_{0 \leq x_1, x_2, x_3 \leq 1} \sum_{i=1}^3 (1 - (1 - x_i^2)) = \sup_{0 \leq x_1, x_2, x_3 \leq 1} \sum_{i=1}^3 (2x_i - x_i^2) \\
 &= 2x - \inf_{0 \leq x_1, x_2, x_3 \leq 1} \sum_{i=1}^3 x_i^2 = 2x - 3 \frac{x^2}{9} = 2x - \frac{x^2}{3}. \quad \square
 \end{aligned}$$

Observe that the inequality $\Delta_2(x) \leq 3$, for every $0 \leq x \leq 4$, is implied by the inequality (10). Putting together the upper bounds in Lemmas 5 and 6 yields Lemma 7:

Lemma 7. *The following inequality holds:*

$$\Delta \leq \frac{38}{9}. \quad (11)$$

From the opposite direction, $\Delta \geq 4$ holds even in a geometric setting, i.e., as implied by several constructions with squares.

Proof. The lower bound $\Delta \geq 4$ is implied by any of the two (obviously different!) configurations shown in Fig. 7.

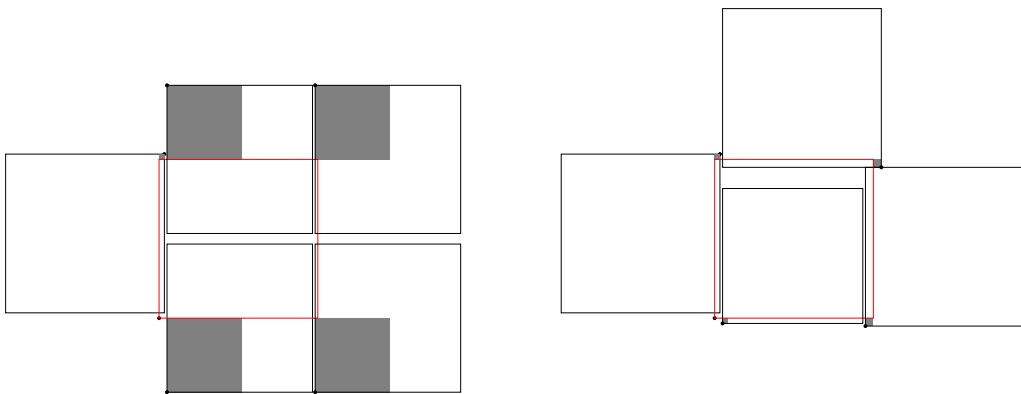


Figure 7: Two constructions with $\Delta \geq 4 - \varepsilon$.

We now prove the upper bound. Recall (4); and by (10) we obtain

$$\Delta_2(4-x) \leq 2(4-x) - \frac{(4-x)^2}{3} = \frac{8+2x-x^2}{3}.$$

We partition the interval $[0, 4]$ into several subintervals, and select the best upper bound we have in each case. We distinguish the following cases (marked with \clubsuit):

\clubsuit $0 \leq x \leq 1$: By (6), $\Delta_1(x) \leq 1$; since $\Delta_2(4-x) \leq 3$, we immediately get that

$$\Delta = \Delta_1(x) + \Delta_2(4-x) \leq 1 + 3 = 4 < 38/9.$$

\clubsuit $1 \leq x \leq 1 + \frac{\sqrt{3}}{3}$: Put $g_1(x) = 14 - 4x + 2x^2$. On this interval we have

$$1 + (x-1)^2 \leq \frac{5}{4} + \frac{(x-1)^2}{4},$$

and so we can use the upper bound in (7) on $\Delta_1(x)$ and obtain

$$\begin{aligned} \Delta(x) &= \Delta_1(x) + \Delta_2(4-x) \leq 1 + (x-1)^2 + \frac{8+2x-x^2}{3} \\ &= \frac{14-4x+2x^2}{3} \leq \frac{g_1(1+1/\sqrt{3})}{3} = \frac{38}{9}. \end{aligned}$$

\clubsuit $1 + \frac{\sqrt{3}}{3} \leq x < 2$: Put $g_2(x) = 50 + 2x - x^2$. On this interval we have

$$1 + (x-1)^2 \geq \frac{5}{4} + \frac{(x-1)^2}{4},$$

and so we can use the upper bound in (8) on $\Delta_1(x)$ and obtain

$$\begin{aligned} \Delta(x) &= \Delta_1(x) + \Delta_2(4-x) \leq \frac{5}{4} + \frac{(x-1)^2}{4} + \frac{8+2x-x^2}{3} \\ &= \frac{50+2x-x^2}{12} \leq \frac{g_2(1+1/\sqrt{3})}{12} = \frac{38}{9}. \end{aligned}$$

\clubsuit $2 \leq x < 3$: Put $g_3(x) = 62 - 4x - x^2$. Using the upper bound in (8) on $\Delta_1(x)$ yields

$$\begin{aligned} \Delta(x) &= \Delta_1(x) + \Delta_2(4-x) \leq \frac{6}{4} + \frac{(x-2)^2}{4} + \frac{8+2x-x^2}{3} \\ &= \frac{62-4x-x^2}{12} \leq \frac{g_3(2)}{12} = \frac{25}{6} < \frac{38}{9}. \end{aligned}$$

\clubsuit $3 \leq x < 4$: Put $g_4(x) = 80 - 10x - x^2$. Using the upper bound in (8) on $\Delta_1(x)$ yields

$$\begin{aligned} \Delta(x) &= \Delta_1(x) + \Delta_2(4-x) \leq \frac{7}{4} + \frac{(x-3)^2}{4} + \frac{8+2x-x^2}{3} \\ &= \frac{80-10x-x^2}{12} \leq \frac{g_4(3)}{12} = \frac{41}{12} < 4 < \frac{38}{9}. \end{aligned}$$

\clubsuit $x = 4$: By (5), $\Delta \leq \Delta_1(x) \leq 2 < 38/9$.

All cases have been checked, and so the proof of the upper bound is complete. \square

Lemma 8. *Each square s_j receives a charge of at most $\frac{47}{9} \text{area}(s_j)$.*

Proof. By Lemma 7, the area-decrease is at most $\frac{38}{9} \text{area}(s_j)$, and so is the charge received by s_j from squares in \mathcal{Q}_1 and from squares in \mathcal{Q}_2 with the exception of the case $i = j$. Adding this last charge yields a total charge of at most $(1 + \frac{38}{9}) \text{area}(s_j) = \frac{47}{9} \text{area}(s_j)$. \square

The combination of Lemmas 3 and 8 now yields the following.

Theorem 4. *Algorithm 1 computes an anchored square packing whose area is at least $9/47$ times the optimal.*

6 Constant-Factor Approximations for Lower-Left Anchored Square Packings

The following greedy algorithm, analogous to Algorithm 1, constructs a lower-left anchored square packing, given a finite point set $P \subset [0, 1]^2$.

Algorithm 2. Set $Q = P$ and $S = \emptyset$. While $Q \neq \emptyset$, repeat the following. For each point $q \in Q$, compute a *candidate* square $s(q)$ such that (i) $s(q) \subseteq [0, 1]^2$ has q as its *lower-left* anchor, (ii) $s(q)$ is empty of points from P in its interior, (iii) $s(q)$ is interior-disjoint from all squares in S , and (iv) $s(q)$ has maximum area. Then choose a largest candidate square $s(q)$, and a corresponding point q , and set $Q \leftarrow Q \setminus \{q\}$ and $S \leftarrow S \cup \{s(q)\}$. When $Q = \emptyset$, return the set of squares S .

Remark. Let ρ_2 denote the approximation ratio of Algorithm 2. The construction in Fig. 8 shows that $\rho_2 \leq 1/3$. Specifically, for $\varepsilon > 0$, with $\varepsilon^{-1} \in \mathbb{N}$, consider the point set $P = \{(\varepsilon, \varepsilon), (0, \frac{1}{2}), (\frac{1}{2}, 0)\} \cup \{(\frac{1}{2} + k\varepsilon, \frac{1}{2} + k\varepsilon) : k = 1, \dots, 1/(2\varepsilon) - 1\}$. Then the area of the packing in Fig. 8(right) is $\frac{3}{4} - O(\varepsilon)$, but Algorithm 2 returns the packing shown in Fig. 8(left) of area $\frac{1}{4} + O(\varepsilon)$.

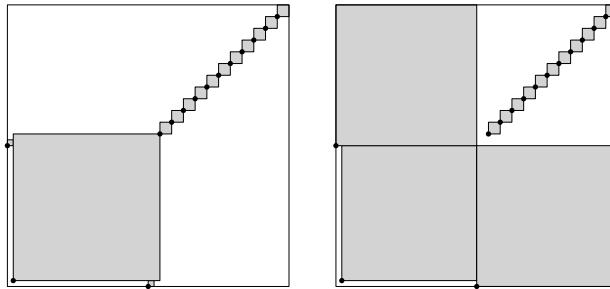


Figure 8: A $1/3$ upper bound for the approximation ratio of Algorithm 2.

We next demonstrate that Algorithm 2 achieves an approximation ratio of $1/3$. According to the above example, this is the best possible for this algorithm.

Theorem 5. *Algorithm 2 computes a lower-left anchored square packing whose area is at least $1/3$ times the optimal.*

Proof. Label the points in $P = \{p_1, \dots, p_n\}$ and the squares in $S = \{s_1, \dots, s_n\}$ in the order in which they are processed by Algorithm 2 with $q = p_i$ and $s_i = s(q)$. Let $G = \sum_{i=1}^n \text{area}(s_i)$ be the area of the greedy packing, and let OPT denote an optimal packing with $A = \text{area}(\text{OPT}) = \sum_{i=1}^n \text{area}(a_i)$, where a_i is the square anchored at p_i .

We charge the area of every optimal square a_i to one or two greedy squares s_ℓ ; and then show that the total area charged to s_ℓ is at most $3 \text{area}(s_\ell)$ for all $\ell = 1, \dots, n$. Consider a square a_i , $1 \leq i \leq n$, with $\text{area}(a_i) > 0$. Let $j = j(i)$ be the minimum index such that s_j intersects the interior of a_i . Let b_i denote the candidate square associated to p_i in step $j + 1$ of Algorithm 2. Note that $b_i \subset a_i$, thus $\text{area}(b_i) < \text{area}(a_i)$. If $\text{area}(b_i) > 0$, then let $k = k(i)$ be the minimum index such that s_k intersects the interior of b_i .

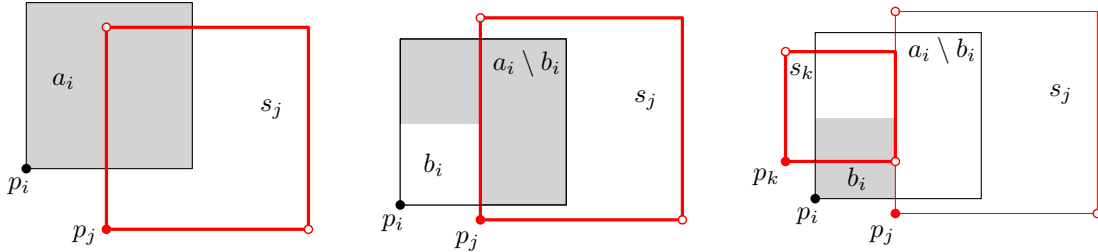


Figure 9: Left: a_i contains the upper-left corner of s_j ; and $\text{area}(a_i)$ is charged to s_j . Middle and Right: a_i contains no corner of s_j , but it contains the lower-right corner of s_k . Then $\text{area}(a_i \setminus b_i)$ is charged to s_j and $\text{area}(b_i)$ is charged to s_k .

We can now describe our *charging scheme*: If a_i contains the upper-left or lower-right corner of s_j , then charge $\text{area}(a_i)$ to s_j (Fig. 9, left). Otherwise, charge $\text{area}(a_i \setminus b_i)$ to s_j , and charge $\text{area}(b_i)$ to s_k (Fig. 9, middle-right).

We first argue that the charging scheme is well-defined, and the total area of a_i is charged to one or two squares (s_j and possibly s_k). Indeed, if no square s_ℓ , $\ell < i$, intersects the interior of a_i , then $a_i \subseteq s_i$, and $j(i) = i$; and if $a_i \not\subseteq s_j$ and no square s_ℓ , $j < \ell < i$, intersects the interior of b_i , then $b_i \subseteq s_i$ and $k(i) = i$.

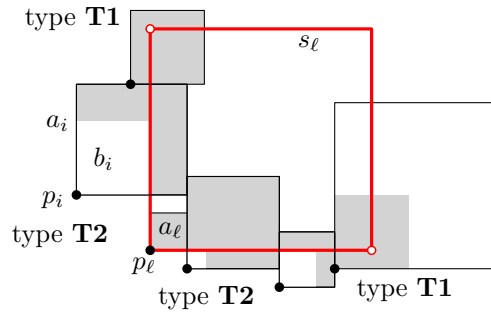


Figure 10: The shaded areas are charged to square s_ℓ .

Note that if $\text{area}(a_i)$ is charged to s_j , then $\text{area}(a_i) \leq \text{area}(s_j)$. Indeed, if $\text{area}(a_i) > \text{area}(s_j)$, then a_i is entirely free at step j , so Algorithm 2 would choose a square at least as large as a_i instead of s_j , which is a contradiction. Analogously, if $\text{area}(b_i)$ is charged to s_k , then $\text{area}(b_i) \leq \text{area}(s_k)$. Moreover, if $\text{area}(b_i)$ is charged to s_k , then the upper-left or lower-right corner of s_k is on the boundary of b_i , and so this corner is contained in a_i ; refer to Fig. 9 (right).

Fix $\ell \in \{1, \dots, n\}$. We show that the total area charged to s_ℓ is at most $3 \text{area}(s_\ell)$. If a square a_i , $i = 1, \dots, n$, sends a positive charge to s_ℓ , then $\ell = j(i)$ or $\ell = k(i)$. We distinguish two types of squares a_i that send a positive charge to s_ℓ ; refer to Fig. 10:

- T1:** a_i contains the upper-left or lower-right corner of s_ℓ in its interior.
- T2:** a_i contains neither the upper-left nor the lower-right corner of s_ℓ .

Since OPT is a packing, at most one optimal square contains each corner of s_ℓ . Consequently, there is at most two squares a_i of type **T1**. Since $\text{area}(a_i) \leq \text{area}(s_\ell)$, the charge received from the squares of type **T1** is at most $2 \text{area}(s_\ell)$.

The following technical lemma is used for controlling the charge that a greedy square s receives from squares of an optimal solution.

Lemma 9. *Let s be an axis-aligned square, and a_1, \dots, a_m be interior-disjoint axis-aligned squares such that*

P1: *each a_i intersects the bottom or the left side of a square s , but*

P2: *the interior of a_i does not contain any corner of s .*

For $i = 1, \dots, m$, let b_i be a maximal square contained in $a_i \setminus \text{int}(s)$. Then $\sum_{i=1}^m \text{area}(a_i \setminus b_i) \leq \text{area}(s)$.

Proof. For $i = 1, \dots, m$, let c_i denote the side-length of a_i , and define the *depth* d_i of a_i as the length of a shortest side of $a_i \cap s$. With this notation, we have $0 \leq d_i \leq c_i \leq c$, the area of the L-shaped region $a_i \setminus b_i$ is

$$\text{area}(a_i \setminus b_i) = \text{area}(a_i) - \text{area}(b_i) = c_i^2 - (c_i - d_i)^2 = 2c_i d_i - d_i^2 = (2c_i - d_i)d_i.$$

We apply a sequence of operations on the squares a_1, \dots, a_m that maintain properties **P1** and **P2**, and monotonically increase $\sum_{i=1}^m (2c_i - d_i)d_i$. The operations successively transform the squares, eventually eliminate $m - 1$ squares, and ensure that the last surviving square is $a_k = s$. This implies $\sum_{i=1}^m 2c_i d_i - d_i^2 \leq \text{area}(s)$, as required.

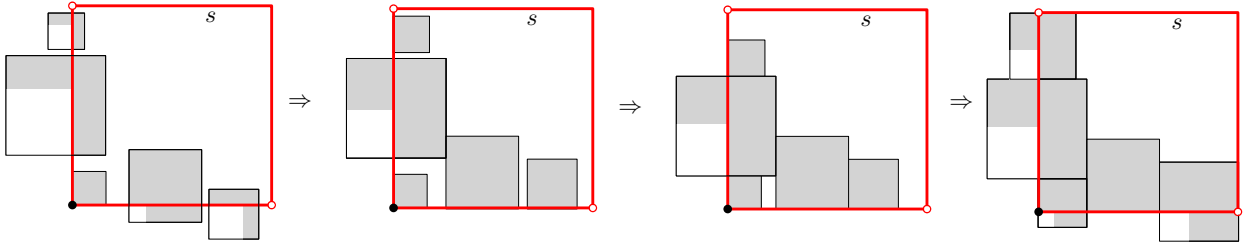


Figure 11: Operations 1, 2 and 3.

Operation 1. Translate a_i horizontally or vertically so that its depth increases until $c_i = d_i$ or a_i is blocked by some other square. See Fig. 11. Perform operation 1 successively for every square a_i in an arbitrary order. The operation can only increase the contribution $c_i^2 - (c_i - d_i)^2$ of a_i since c_i is fixed and the depth d_i can only increase.

Operation 2. Translate a_i horizontally or vertically so that its depth remains fixed but a_i moves closer to the lower-left corner p until it reaches p or is blocked by another square. See Fig. 11. Perform operation 2 successively for every square a_i in the order determined by their distance from p . This operation does not change the contribution of the squares a_i .

Operation 3. If a_i intersects the bottom (left) side of s and its right (top) side is not in contact with any other square or a corner of s , then dilate a_i from its upper-left (lower-right) corner until it is blocked by another square or the boundary of s . See Fig. 11 (right).

Since operation 3 increases c_i and keeps d_i fixed, the contribution $(2c_i - d_i)d_i$ of a_i increases. Note that if the lower-left corner of a_i is p originally and operation 3 is applied to a_i , then its lower-left corner may move to the exterior of s .

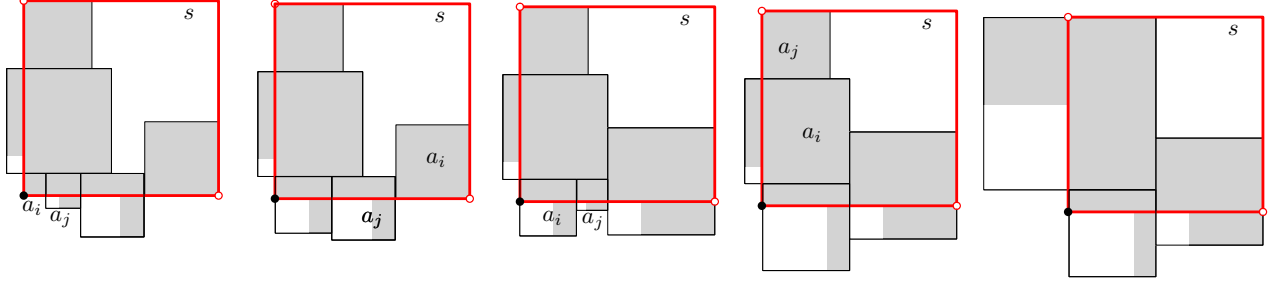


Figure 12: Operation 4.

Operation 4. Consider two squares, a_i and a_j , such that both intersect the bottom side of s and they are adjacent. Without loss of generality, assume a_i is left of a_j and $d_i \geq d_j$. See Fig. 12. We wish to increase c_i and decrease c_j while $c_i + c_j$ is fixed, and such that their depths does not increase. We distinguish two cases to ensure that a_i intersects the bottom side of s after the operation. If $d_j < c_j$, then dilate a_i from its upper-left corner and a_j from its upper-right corner simultaneously until $d_j = c_j$ or a_i is blocked by some other square or a corner of s . In this case, d_i and d_j remain constant, and the contribution of a_i and a_j increases or remains the same. Indeed, if c_i increases and c_j decreases by $\varepsilon > 0$, then

$$\begin{aligned} (2(c_i + \varepsilon) - d_i)d_i + (2(c_j - \varepsilon) - d_j)d_j &= (2c_i - d_i)d_i + (2c_j - d_j)d_j + 2\varepsilon(d_i - d_j) \\ &\geq (2c_i - d_i)d_i + (2c_j - d_j)d_j. \end{aligned}$$

If $d_j = c_j$, then dilate a_i from its upper-left corner and a_j from its lower-right corner simultaneously until $c_j = 0$ or a_i is blocked by some other square or a corner of s . In this case, d_i is constant, and $d_j = c_j$ decreases. The contribution of a_i and a_j increases or remains the same:

$$\begin{aligned} (2(c_i + \varepsilon) - d_i)d_i + (d_j - \varepsilon)^2 &= (2c_i - d_i)d_i + d_j^2 + 2\varepsilon(d_i - d_j) + \varepsilon^2 \\ &\geq (2c_i - d_i)d_i + d_j^2 + 2\varepsilon(d_i - d_j). \end{aligned}$$

We apply analogous operations to adjacent squares intersecting the left side of s .

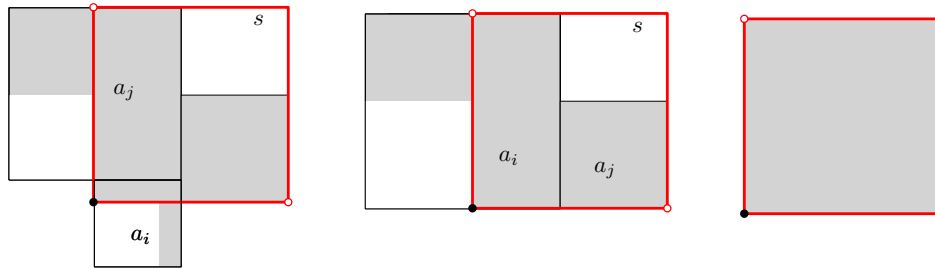


Figure 13: Operation 5.

Operation 5. Consider two adjacent squares, a_i and a_j , such that a_i is incident to the lower-left corner of s and intersects the bottom (resp., left) side of s , and a_j intersects the left (resp., bottom) side of s . See Fig. 13. Note that $d_j \leq c_i$, otherwise operation 2 could increase c_i ; and if $d_j < c_i$, then $c_j = d_j$, otherwise operation 1 could increase d_j . Let R be the axis-aligned bounding box of $(a_i \cup a_j) \cap s$, which is incident to the lower-left corner of p and $\text{area}(R) = c_i(d_i + c_j)$. We replace a_i and a_j with one new square a_k such that $a_k \cap s = R$.

It remains to show that operation 5 increases the total contribution of the squares. We distinguish two cases depending on which side of R is longer.

Case 1. If $d_i + c_j \geq c_i$, then a_k has side-length $c_k = d_i + c_j$ and depth $d_k = c_i$. In case $d_j = c_i$ (Fig. 12, left), we have

$$\begin{aligned}
(2c_i - d_i)d_i + (2c_j - d_j)d_j &= (2c_i - d_i)d_i + (2c_j - c_i)c_i \\
&= 2c_id_i - d_i^2 + 2c_jc_i - c_i^2 \\
&\leq 2c_id_i + 2c_ic_j - c_i^2 \\
&= (2(d_i + c_j) - c_i)c_i \\
&= (2c_k - d_k)d_k.
\end{aligned}$$

Otherwise we have $d_j = c_j < c_i$ and $d_j = c_j$ (Fig. 12, right). Thus,

$$\begin{aligned}
(2c_i - d_i)d_i + c_j^2 &= 2c_id_i + c_j^2 - d_i^2 \\
&\leq 2c_id_i + c_j^2 - (c_i - c_j)^2 \\
&= 2c_id_i + 2c_ic_j - c_i^2 \\
&= (2(d_i + c_j) - c_i)c_i \\
&= (2c_k - d_k)d_k.
\end{aligned}$$

Case 2. If $d_i + c_j \leq c_i$, then a_k has side-length $c_k = c_i$ and depth $d_k = d_i + c_j$. In this case, we have $d_j = c_i$, otherwise operation 1 could increase d_j . Thus,

$$\begin{aligned}
(2c_i - d_i)d_i + c_j^2 &= 2c_id_i + c_j^2 - d_i^2 \\
&= 2c_id_i + 2c_j^2 - (c_j^2 + d_i^2) \\
&= 2c_id_i + 2c_ic_j - 2(c_i - c_j)c_j - (d_i^2 + c_j^2) \\
&\leq 2c_id_i + 2c_ic_j - 2d_ic_j - (d_i^2 + c_j^2) \\
&= 2c_id_i + 2c_ic_j - (d_i + c_j)^2 \\
&= (2c_i - (d_i + c_j))(d_i + c_j) \\
&= (2c_k - d_k)d_k.
\end{aligned}$$

After successively applying operation 5, we obtain a single square $a_k = s$, whose contribution is $\text{area}(a_k) = \text{area}(s)$, as required. \square

By Lemma 9, s_ℓ receives a charge of at most $\text{area}(s_\ell)$ from squares of type **T2**. It follows that each s_ℓ received a charge of at most $3 \text{area}(s_\ell)$. Consequently,

$$A = \text{area}(\text{OPT}) = \sum_{i=1}^n \text{area}(a_i) \leq 3 \sum_{\ell=1}^n \text{area}(s_\ell) = 3G,$$

and thus $G \geq A/3$. This completes the proof of Theorem 5. \square

7 Approximation Schemes

In this section, we show that there is a PTAS (resp., QPTAS) for the maximum anchored square (resp., rectangle) problem using reductions to the minimum weight independent set (MWIS) problem for axis-aligned squares (resp., rectangles) in the plane. The current best approximation ratio

for MWIS with axis-aligned rectangles is $O(\log n / \log \log n)$ [18], and $O(\log \log n)$ in the unweighted case [16]. The first PTAS for MWIS with axis-aligned squares was given by Erlebach et al. [25]; the running time was later improved by Chan [17]. Specifically, for n axis-aligned squares an $(1 - \varepsilon)$ -approximation for MWIS can be computed in time $n^{O(1/\varepsilon)}$ [17].

For MWIS with axis-aligned rectangles, Adamaszek and Wiese [3] obtained a QPTAS. Specifically, for n axis-aligned rectangles an $(1 - \varepsilon)$ -approximation for MWIS can be computed in time $\exp(\text{poly}(\varepsilon^{-1} \log n))$ [3]. For the unweighted case, the running time was subsequently improved to $\exp(\log n \cdot \text{poly}(\varepsilon^{-1} \log \log n))$ by Chuzhoy and Ene [20]. Extending the seminal result by Adamaszek and Wiese, a QPTAS was also obtained for MWIS with polygons in the plane: For n axis-aligned polygons, each with $O(1)$ vertices, an $(1 - \varepsilon)$ -approximation for MWIS can be computed in time $\exp(\text{poly}(\varepsilon^{-1} \log n))$ [2, 4, 28].

7.1 Approximation Schemes for Anchored Packings

In this section we reduce the anchored rectangle (resp., square) packing problem to the MWIS for axis-aligned rectangles (resp., squares) in the plane. The reduction is easier for the case of rectangles, since the anchored rectangle packing problem can be discretized.

Anchored rectangles. We reduce the anchored rectangle packing problem to MWIS with axis-aligned rectangles, for which a QPTAS is available [3]. We obtain a QPTAS for the maximum area anchored rectangle packing problem.

Theorem 6. *Given a set of n points in $[0, 1]^2$ and any $\varepsilon > 0$, there is an algorithm (QPTAS) that computes an anchored rectangle packing of area at least $(1 - \varepsilon)\text{area}(\text{OPT})$ in time $\exp(\text{poly}(\varepsilon^{-1} \log n))$.*

Proof. Let P be a set of n points in U . Consider the Hanan grid [27] induced by the points, i.e., the union of ∂U and the at most $2n$ horizontal and vertical unit segments incident to the n points and contained in U . By [10, Proposition 1], there exists an anchored rectangle packing of maximum area OPT such that all corners of the rectangle are vertices of the Hanan grid. For every $p \in P$, let $C(p)$ be the set of empty rectangles anchored at p such that all four corners are vertices of this grid; and let $C = \bigcup_{p \in P} C(p)$. Since the Hanan grid has $O(n^2)$ vertices and each rectangle in C is spanned by a point in P and a vertex of the grid, we have $|C| = O(n^3)$.

Let δ ($0 < \delta \leq \varepsilon$) be less than half of the minimum difference between distinct x - and y -coordinates of the points in P . For every $p \in P$, deform each rectangle $r \in C(p)$ into an octagon \hat{r} as follows: First shrink r by translating all four sides toward the center of r by $\delta/2$, and then add a square of side length δ centered at p . Let $\text{area}(r)$ be the weight of the octagon \hat{r} . Denote by $\hat{C}(p)$ the resulting set of octagons \hat{r} , and let $\hat{C} = \bigcup_{p \in P} \hat{C}(p)$. Note that two octagons in \hat{C} intersect if and only if the corresponding rectangles in C overlap or have the same anchor.

Let $\varepsilon > 0$ be given. Using the QPTAS for MWIS with polygons [2, 4, 28], an $(1 - \varepsilon)$ -approximation of a MWIS for the set \hat{C} of $m = O(n^3)$ weighted octagons can be computed in time $\exp(\text{poly}(\varepsilon^{-1} \log m)) = \exp(\text{poly}(\varepsilon^{-1} \log n))$. Reverse the deformations for the octagons returned by the MWIS algorithm, and return this set of rectangles.

By construction, the rectangles returned by the algorithm are contained in U and anchored at points in P . At most one rectangle is anchored at each $p \in P$, due to the δ -deformations. The rectangles are interior-disjoint due to the choice of δ . That is, the algorithm returns an anchored rectangle packing of area at least $(1 - \varepsilon)\text{OPT}$, as claimed. \square

Lower-left anchored rectangles. The proof of Theorem 6 carries over verbatim to lower-left anchored rectangle packings. We obtain a QPTAS for the maximum area anchored rectangle packing problem.

Corollary 1. *Given a set of n points in $[0, 1]^2$ and any $\varepsilon > 0$, there is an algorithm (QPTAS) that computes a lower-left anchored rectangle packing of area at least $(1 - \varepsilon) \text{area}(\text{OPT})$ in time $\exp(\text{poly}(\varepsilon^{-1} \log n))$.*

Anchored squares. It is worth noting that no similar discretization is known for the anchored square packing problem. With an ε -factor loss of the total area, however, we can construct $O(n \log(n/\varepsilon))$ squares for which MWIS gives an anchored packing of area at least $(1 - \varepsilon) \text{area}(\text{OPT})$. After a similar transformation ensuring that the set of squares anchored at the same point contain the point in their interiors, and under the assumption that the points are in general position, we obtain a PTAS for the maximum-area anchored square packing problem.

Theorem 7. *There is a PTAS for the maximum area anchored square packing problem. For every $\varepsilon > 0$, there is an algorithm that computes, for a set of n points in general position in $[0, 1]^2$, an anchored square packing of area at least $(1 - \varepsilon) \text{area}(\text{OPT})$ in time $n^{O(1/\varepsilon)}$.*

Proof. First we show that the problem can be discretized if we are willing to lose a factor $\varepsilon > 0$ of the maximum area of a packing. Let P be a set of n points in $[0, 1]^2$, and let OPT be an anchored square packing of maximum area. For a given $\varepsilon > 0$, drop all squares in OPT of area less than ε/n , and shrink all remaining squares in OPT so that its anchor remains the same, and its area decreases by a factor of at most $1 + \varepsilon$ to $(\varepsilon/n)(1 + \varepsilon)^k$ for some integer $k \geq 0$. Denote by OPT_ε the resulting anchored square packing. We have dropped small squares of total area at most ε , and the area of any remaining square decreased by a factor of at most $1 + \varepsilon$. Consequently, $(\text{area}(\text{OPT}) - \varepsilon)/(1 + \varepsilon) \leq \text{area}(\text{OPT}_\varepsilon)$. For anchored square packings, we have $\text{area}(\text{OPT}) \geq 5/32$ by Theorem 2, which yields

$$\left(1 - \frac{32\varepsilon}{5}\right) \text{area}(\text{OPT}) \leq \text{area}(\text{OPT}_\varepsilon). \quad (12)$$

Let $\varepsilon > 0$ be given. For every set P of n points in general position in $[0, 1]^2$, we compute a set \widehat{C} of $O((n/\varepsilon) \log(n/\varepsilon))$ candidate squares, and show that a MWIS of \widehat{C} (where the weight of each square is its area) has area at least $(1 - \varepsilon) \text{area}(\text{OPT})$. For each point $p \in P$, consider a set $C(p)$ of empty squares anchored at p and contained in U whose areas are of the form $(\varepsilon/n)(1 + \varepsilon)^k$, for $k = 0, \dots, \lfloor \log(n/\varepsilon)/\log(1 + \varepsilon) \rfloor = O(\varepsilon^{-1} \log(n/\varepsilon))$. Let $C(p) = \bigcup_{p \in P} C(p)$.

Translate each square in $C(p)$, for all $p \in P$, by a sufficiently small δ so that each square contains its anchor in its interior, but does not intersect any new horizontal or vertical line through the corners of any square in $\bigcup_{p \in P} C(p)$. Denote by $\widehat{C}(p)$ the resulting set of squares. Let $\widehat{C} = \bigcup_{p \in P} \widehat{C}(p)$. Note that the squares in $\widehat{C}(p)$ pairwise intersect for each $p \in P$, hence an MWIS contains precisely one square (possibly of 0 area) for each point in P . Denote by OPT_{MWIS} a MWIS for the set \widehat{C} . Since each square in OPT_ε is contained in some square in C , and the translations do not change the areas, we have

$$\text{area}(\text{OPT}_\varepsilon) \leq \text{area}(\text{OPT}_{\text{MWIS}}). \quad (13)$$

The number of squares in \widehat{C} is

$$|\widehat{C}| = \sum_{p \in P} |\widehat{C}(p)| = n \cdot O(\varepsilon^{-1} \log(n/\varepsilon)) = O((n/\varepsilon) \log(n/\varepsilon)).$$

Using the PTAS for MWIS with axis-aligned squares due to Chan [17], an $(1 - \varepsilon)$ -approximation for the set \widehat{C} of $m = O((n/\varepsilon) \log(n/\varepsilon))$ weighted axis-aligned squares can be computed in time $m^{O(1/\varepsilon)} = n^{O(1/\varepsilon)}$. Finally, for the set of squares returned by Chan's algorithm, reverse the translation by δ , and denote by ALG_ε the resulting set of squares in C .

By construction, the squares in ALG_ε are contained in U and anchored at points in P . At most one square is anchored at each $p \in P$, due to the δ -translations, but the squares are interior-disjoint due to the choice of δ . That is, ALG_ε is an anchored square packing. Since ALG_ε is $(1 - \varepsilon)$ -approximation for MWIS where the weights are the areas, we have

$$(1 - \varepsilon) \text{area}(\text{OPT}_{\text{MWIS}}) \leq \text{area}(\text{ALG}_\varepsilon). \quad (14)$$

The combination of (12), (13), and (14) yields

$$\text{area}(\text{ALG}_\varepsilon) \geq \left(1 - \frac{32\varepsilon}{5}\right) (1 - \varepsilon) \text{OPT} > \left(1 - \frac{37\varepsilon}{5}\right) \text{OPT}.$$

By redefining ε we obtain the claimed result. \square

Remark. The proof of Theorem 7 does not immediately carry over to *lower-left* anchored square packing. The maximum area of a lower-left anchored square packing can be arbitrarily small (i.e., there is no constant lower-bound guarantee), e.g., when the points in P are close to the top or right boundary of U , or when n is large and the points are placed along the diagonal of unit slope. Dropping squares of area less than ε/n may have a significant impact, and (12) no longer holds. In Section 7.2, we introduce a new concept (called “reach”) that leads to a PTAS for lower-left anchored squares.

7.2 The Reach of Anchored Packings

In this section, we show that a greedy packing covers a constant fraction of the region that could possibly be covered by some lower-left anchored rectangles (resp., squares). We then employ these bounds in the design of a PTAS for lower-left anchored squares.

The reach of lower-left anchored squares. We define the *reach of lower-left anchored squares* as $W_{\text{sq}}(P) = \bigcup_{p \in P} s(p)$, where $s(p)$ is the (unique) maximal square contained in U whose lower-left corner is p and that is empty of points of P in its interior. Observe that $W_{\text{sq}}(P)$ is the union of candidate squares in the first iteration of Algorithm 2. For a square s and a parameter $\lambda > 0$, let λs denote the square homothetic to s and concentric to s of ratio λ .

Lemma 10. *Let $P = \{p_1, \dots, p_n\} \subset U$ be a point set with a lower-left anchored square packing $S = \{s_1, \dots, s_n\}$ returned by Algorithm 2. Then $W_{\text{sq}}(P) \subset \bigcup_{i=1}^n 3s_i$.*

Proof. We need to show that every point $q \in W_{\text{sq}}(P)$ lies in $\bigcup_{i=1}^n 3s_i$. This claim clearly holds when q is covered by one of the squares s_1, \dots, s_n .

Let $q \in W_{\text{sq}}(P) \setminus P$ be a point left uncovered by the greedy square packing. By the definition of $W_{\text{sq}}(P)$, point q lies in some lower-left anchored square $s = s(p_i)$, $p_i \in P$, of maximum area contained in $[0, 1]^2$ but empty of points of P in its interior. Note that the boundary of s contains at least one point from P , namely $p_i \in P$.

Let s_j be the first square chosen by Algorithm 2 that intersects the interior of s . We are guaranteed the existence of s_j because if no previous greedy square intersects the interior of s , then $s_i = s$ (possibly $i = j$). At the beginning of the step j of Algorithm 2, no square in S intersects s .

By the greedy choice, we have $\text{area}(s_j) \geq \text{area}(s)$. Since $s_j \cap s \neq \emptyset$ and the side-length of s is less than or equal to the side-length of s_j , we have $q \in s \subset 3s_j$. \square

Theorem 8. *For every finite point set P in $[0, 1]^2$, there is a lower-left anchored square packing of total area at least $\frac{1}{9} \text{area}(W_{\text{sq}}(P))$.*

Proof. Given a point set $P = \{p_1, \dots, p_n\} \subset [0, 1]^2$ and a greedy lower-left anchored square packing $S = \{s_1, \dots, s_n\}$, Lemma 10 yields

$$\text{area}(W_{\text{sq}}(P)) \leq \text{area}\left(\bigcup_{i=1}^n 3s_i\right) \leq \sum_{i=1}^n \text{area}(3s_i) = 9 \sum_{i=1}^n \text{area}(s_i),$$

which implies $\sum_i^n \text{area}(s_i) \geq \frac{1}{9} \text{area}(W_{\text{sq}}(P))$, as claimed. \square

Approximation schemes for lower-left square anchored packings. Even though the area of lower-left anchored square packings is not bounded by any constant, and Theorem 2 no longer holds, the PTAS in Theorem 7 can be extended for this variant. By definition, $W_{\text{sq}}(P)$ contains all squares of a lower-left anchored square packing for P , and we have $\text{area}(\text{OPT}) \geq \frac{1}{9} \text{area}(W_{\text{sq}}(P))$ by Theorem 8. By adjusting the resolution of the candidate squares, we obtain a PTAS for the maximum area lower-left anchored square packing problem. Instead of dropping all squares of area less than ε/n , as in the proof of Theorem 7, we drop all squares of area less than $\varepsilon \text{area}(W_{\text{sq}}(P))/n$.

Corollary 2. *There is a PTAS for the maximum area anchored square packing problem. For every $\varepsilon > 0$, there is an algorithm that computes, for a set of n points in general position in $[0, 1]^2$, a lower-left anchored square packing of area at least $(1 - \varepsilon) \text{area}(\text{OPT})$ in time $n^{O(1/\varepsilon)}$.*

Remark. We can analogously define the reach of lower-left anchored rectangles and the reach of anchored squares. While they are not needed for our approximation schemes, for the sake of completeness we derive lower bounds for their areas, as well.

The reach of lower-left anchored rectangles. For a set P of n points in U , we define the reach of lower-left anchored rectangles as $W(P) = \bigcup_{p \in P} w(p)$, where $w(p)$ is the union of all axis-aligned empty rectangles in U whose lower-left corner is p .

Observation 2. *For a set P of n points in U ,*

$$W(P) = \bigcup_{p \in P} r(p),$$

where $r(p)$ is the axis-aligned rectangle spanned by p and the upper-right corner of U .

Proof. By definition, $w(p) \subseteq r(p)$ for all $p \in P$, which readily implies $W(P) \subseteq \bigcup_{p \in P} r(p)$. It remains to prove $\bigcup_{p \in P} r(p) \subseteq W(P)$. Let $q \in \bigcup_{p \in P} r(p)$. Then there is a point $p \in P$ for which $q \in r(p)$. Let $p \in P$ such that $q \in r(p)$ and $x(p) + y(p)$ is minimal. Then the rectangle spanned by p and q is empty, hence it is contained in $w(p)$, and so $q \in w(p)$. Consequently $q \in W(P)$, and $\bigcup_{p \in P} r(p) \subseteq W(P)$, as required. \square

By Observation 2, $W(P)$ is a simple orthogonal polygon in which every lower-left corner is an element in P . Vertical lines through the lower-left corners of $W(P)$ subdivide $W(P)$ into axis-aligned rectangles whose lower left-corners are in P . According to the main result in [24], every finite point set P_r in an axis-aligned rectangle r , where the lower-left-corner of r is in P_r , admits a lower-left anchored rectangle packing within r whose area is at least $0.091 \text{area}(r)$. Consequently, $\text{area}(\text{OPT}) \geq 0.091 \text{area}(W(P))$.

The reach of anchored squares. For a finite (nonempty) set $P \subset [0, 1]^2$, the *reach of anchored squares* $R_{\text{sq}}(P)$ is the union of all maximal empty squares anchored at a point in P and contained in U . Obviously, we have $R_{\text{sq}}(P) \geq 5/32$ by Theorem 2, and we suspect that $\text{area}(R_{\text{sq}}(P)) \geq \frac{1}{2}$ for every P . If true, this bound would be the best possible: choose a small $\varepsilon > 0$ and place all n points in the ε -neighborhood of $(\frac{1}{2}, 0)$ on the lower side of U . The side-length of any anchored square is at most $\frac{1}{2} + \varepsilon$, which is an upper bound for the distance of these points from the left and right side of $[0, 1]^2$. Consequently all anchored squares lie below the horizontal line $y = \frac{1}{2} + \varepsilon$, and so the area of their reach is at most $\frac{1}{2} + \varepsilon$.

8 Conclusion

We conclude with a few open problems:

1. Is the problem of computing the maximum-area anchored rectangle (respectively, square) packing NP-hard?
2. Is there a polynomial-time approximation scheme for the problem of computing an anchored rectangle packing of maximum area?
3. What lower bound on $A(n)$ can be obtained by extending Lemma 1 concerning rectangles from 2 to 3 points? Is there a short proof of Lemma 1? (While such an extension would probably not allow reaching the upper bound of $2/3$ asymptotically, it would likely bring a notable improvement.)
4. Does Algorithm 1 for computing an anchored square packing of maximum area achieve a ratio of $1/4$? By Theorem 4 and the construction in Fig. 5, the approximation ratio is between $9/47 = 1/5.22\dots$ and $1/4$. Improvements beyond the ratio $1/5$ are particularly exciting.
5. Is the reach of anchored squares always at least $1/2$? (Does $\text{area}(R_{\text{sq}}(P)) \geq \frac{1}{2}$ hold for every nonempty set P ?)
6. Is $A(n) = \frac{2}{3}$? Is $A_{\text{sq}}(n) = \frac{7}{27}$?
7. What upper and lower bounds on $A(n)$ and $A_{\text{sq}}(n)$ can be established in higher dimensions?
8. A natural variant of anchored squares is one where the anchors must be the centers of the squares. What approximation can be obtained in this case?

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A Proof of Lemma 1

Let $P = \{p_1, p_2\}$ with $p_i = (x_i, y_i)$ for $i \in \{1, 2\}$. We can assume that $R = U = [0, 1]^2$, $y_1 = 0$, and further that $x_1 \leq x_2$ by applying a reflection with respect to a vertical line if necessary. Consider the grid induced by the two points in U , consisting of six rectangles r_i , $i = 1, \dots, 6$, as shown in Fig. 14. Put $a_i = \text{area}(r_i)$, $i = 1, \dots, 6$. As a shorthand notation, a union of rectangles, $r_{i_1} \cup \dots \cup r_{i_k}$ is denoted $r_{i_1 \dots i_k}$, and $a_{i_1 \dots i_k} = \text{area}(r_{i_1 \dots i_k}) = a_{i_1} + \dots + a_{i_k}$ denotes its area.

It is not hard to show that every maximal rectangle packing for $P = \{p_1, p_2\}$ is one of the eight packings in Fig. 14 (although, we do not use this fact in the remainder of the proof). Specifically, p_1 is a lower-left or lower-right corner of a rectangle. If p_1 is a lower-right corner, by maximality the rectangle has unit height; then p_2 can be the anchor of any of r_2 , r_3 , r_5 , and r_6 , producing the packings r_{124} , r_{134} , r_{145} , and r_{146} . If p_1 is the lower-left corner of a rectangle of height 1, then p_2 can be the anchor of r_3 or r_6 , producing r_{235} and r_{256} ; if p_1 is the lower-left corner of a rectangle of height y_2 , then p_2 can be the anchor of r_{12} or r_3 , which yields the packings r_{1256} and r_{356} .

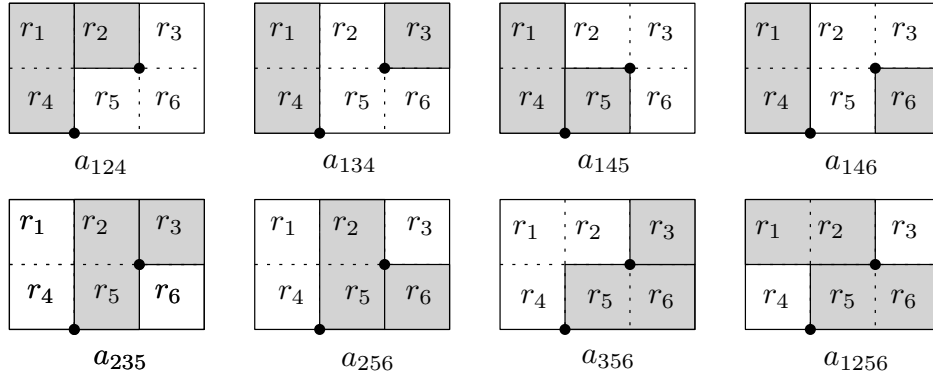


Figure 14: Eight possible anchored rectangle packings for p_1 and p_2 .

The area of each of the eight packing in Fig. 14 is a quadratic function in the variables x_1, x_2, y_2 over the domain

$$D = \{(x_1, x_2, y_2) \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq 1 \text{ and } 0 \leq y_2 \leq 1\}.$$

We wish to find the minimum A of the upper envelope of these eight 3-variate functions over the domain D . Observe that

$$a_{1256} + a_{134} = 1 + a_1. \quad (15)$$

In particular, if $a_1 \geq 1/6$ holds, the desired lower bound, $7/12$, follows. For the case analysis below, we also use the following fact.

Fact 1. Consider the quadratic equation $12\tau^2 - 7\tau + 1 = 0$. Since its roots are $\tau_1 = 1/4$ and $\tau_2 = 1/3$, we have $12\tau^2 - 7\tau + 1 \leq 0$ for $\tau \in [\tau_1, \tau_2]$.

Case analysis. We distinguish six cases covering the range of x_1 : $0 \leq x_1 \leq 1/2$.

Case 1: $x_1 \leq 1/4$. If $y_2 \geq 1/3$ then

$$\max(a_{356}, a_{1256}) \geq (1 - x_1)y_2 + \frac{1}{2}(1 - y_2) \geq \frac{3}{4}y_2 + \frac{1}{2}(1 - y_2) = \frac{1}{2} + \frac{y_2}{4} \geq \frac{1}{2} + \frac{1}{12} = \frac{7}{12},$$

as required. If $y_2 \leq 1/3$, we distinguish two subcases:

Case 1.1: $x_2 \leq 1/2$. Then select rectangle r_3 anchored at p_2 , and one of r_{14} and r_{25} anchored at p_1 . It follows that

$$\max(a_{134}, a_{235}) \geq \frac{x_2}{2} + \frac{2}{3}(1 - x_2) = \frac{2}{3} - \frac{x_2}{6} \geq \frac{2}{3} - \frac{1}{12} = \frac{7}{12},$$

as required.

Case 1.2: $x_2 \geq 1/2$. Then

$$a_{235} = (1 - x_1) - a_6 = (1 - x_1) - (1 - x_2)y_2 \geq \frac{3}{4} - \frac{1}{2} \cdot \frac{1}{3} = \frac{7}{12},$$

as required.

Case 2: $1/4 \leq x_1 \leq 1/3$. If $y_2 \geq 1/2$, we have

$$\max(a_{356}, a_{1256}) \geq (1 - x_1)y_2 + \frac{1}{2}(1 - y_2) \geq \frac{2}{3}y_2 + \frac{1}{2}(1 - y_2) = \frac{1}{2} + \frac{y_2}{6} \geq \frac{1}{2} + \frac{1}{12} = \frac{7}{12},$$

as required.

If $y_2 \leq 1/3$, we have

$$a_1 = x_1(1 - y_2) \geq \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6},$$

and the lower bound follows from (15).

If $1/3 \leq y_2 \leq 1/2$, we distinguish two subcases:

Case 2.1: $x_2 \leq 1/2$.

If $\frac{1}{3} \leq y_2 \leq \frac{1}{6(1-2x_1)}$, then

$$a_1 = x_1(1 - y_2) \geq x_1 \left(1 - \frac{1}{6(1-2x_1)}\right) = \frac{x_1(5 - 12x_1)}{6(1-2x_1)} \geq \frac{1}{6},$$

and the lower bound follows from (15) (due to Fact 1).

If $\frac{1}{6(1-2x_1)} \leq y_2 \leq \frac{1}{2}$, then

$$\begin{aligned} a_{356} &= (1 - x_2) + (x_2 - x_1)y_2 \geq (1 - x_2) + (x_2 - x_1)\frac{1}{6(1-2x_1)} \\ &= \frac{6 - 13x_1 - 5x_2 + 12x_1x_2}{6(1-2x_1)} \geq \frac{7}{12}, \end{aligned}$$

as required. The last inequality above is equivalent to

$$5(1 - 2x_2) \geq 12x_1(1 - 2x_2),$$

which holds for $x_1 \leq 5/12$.

Case 2.2: $x_2 \geq 1/2$.

If $\frac{1}{3} \leq y_2 \leq 1 - \frac{1}{6x_1}$, then

$$a_1 = x_1(1 - y_2) \geq x_1 \frac{1}{6x_1} = \frac{1}{6},$$

and the lower bound follows from (15).

If $1 - \frac{1}{6x_1} \leq y_2 \leq \frac{1}{2}$, then

$$\begin{aligned} a_{1256} &= (1 - x_1)y_2 + x_2(1 - y_2) = x_2 + y_2(1 - x_1 - x_2) \geq x_2 + \frac{6x_1 - 1}{6x_1}(1 - x_1 - x_2) \\ &= \frac{6x_1x_2 + 6x_1 - 6x_1^2 - 6x_1x_2 - 1 + x_1 + x_2}{6x_1} \geq \frac{7x_1 - 6x_1^2 - 1/2}{6x_1} \geq \frac{7}{12}, \end{aligned}$$

as required. Indeed, the last inequality above is equivalent to $12x_1^2 - 7x_1 + 1 \leq 0$, which holds since $x_1 \in [1/4, 1/3]$ and according to Fact 1.

Case 3: $1/3 \leq x_1 \leq 2/5$. If $y_2 \leq 1/2$, then

$$a_1 = x_1(1 - y_2) \geq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

and the lower bound follows from (15). If $y_2 \geq 1/2$, we distinguish two subcases:

Case 3.1: $x_2 \leq 3x_1/2$. Then

$$\begin{aligned} a_{146} &= x_1 + (1 - x_2)y_2 \geq x_1 + \left(1 - \frac{3x_1}{2}\right)y_2 \\ &\geq x_1 + \frac{1}{2} \left(1 - \frac{3x_1}{2}\right) = \frac{1}{2} + \frac{x_1}{4} \geq \frac{1}{2} + \frac{1}{12} = \frac{7}{12}, \end{aligned}$$

as required.

Case 3.2: $x_2 \geq 3x_1/2$. Then

$$\begin{aligned} a_{1256} &= (1 - x_1)y_2 + x_2(1 - y_2) \geq (1 - x_1)y_2 + \frac{3x_1}{2}(1 - y_2) \\ &= y_2 + \frac{3x_1}{2} - \frac{5x_1y_2}{2} = \frac{3x_1}{2} + y_2 \left(1 - \frac{5x_1}{2}\right) \\ &\geq \frac{3x_1}{2} + \frac{1}{2} \left(1 - \frac{5x_1}{2}\right) = \frac{1}{2} + \frac{x_1}{4} \geq \frac{1}{2} + \frac{1}{12} = \frac{7}{12}, \end{aligned}$$

as required.

Case 4: $2/5 \leq x_1 \leq 5/12$. Observe that $x_1 \geq 2/5 > 7/19$, and consequently, $19x_1/7 - 1 > 0$, as used below in Case 4.2. If $y_2 \leq 7/12$, then

$$a_1 = x_1(1 - y_2) \geq \frac{2}{5} \cdot \frac{5}{12} = \frac{1}{6},$$

and the lower bound follows from (15). If $y_2 \geq 7/12$, we distinguish two subcases:

Case 4.1: $x_2 \leq 12x_1/7$. Then

$$\begin{aligned} a_{146} &= x_1 + (1 - x_2)y_2 \geq x_1 + \left(1 - \frac{12x_1}{7}\right)y_2 \\ &\geq x_1 + \frac{7}{12} \left(1 - \frac{12x_1}{7}\right) = \frac{7}{12}, \end{aligned}$$

as required.

Case 4.2: $x_2 \geq 12x_1/7$. Then

$$\begin{aligned} a_{1256} &= (1 - x_1)y_2 + x_2(1 - y_2) \geq (1 - x_1)y_2 + \frac{12x_1}{7}(1 - y_2) \\ &= \frac{12x_1}{7} - y_2 \left(\frac{19x_1}{7} - 1 \right) \geq \frac{12x_1}{7} - \left(\frac{19x_1}{7} - 1 \right) \\ &= 1 - x_1 \geq \frac{7}{12}, \end{aligned}$$

as required.

Case 5: $5/12 \leq x_1 \leq 4/9$. If $y_2 \leq 3/5$, then

$$a_1 = x_1(1 - y_2) \geq \frac{5}{12} \cdot \frac{2}{5} = \frac{1}{6},$$

and the lower bound follows from (15). If $y_2 \geq 3/5$, we distinguish two subcases:

Case 5.1: $x_2 \leq 13/18$. Then

$$a_{146} = x_1 + (1 - x_2)y_2 \geq \frac{5}{12} + \frac{5}{18} \cdot \frac{3}{5} = \frac{7}{12},$$

as required.

Case 5.2: $x_2 \geq 13/18$. If $y_2 \leq 5/6$, then

$$a_{1256} = (1 - x_1)y_2 + x_2(1 - y_2) \geq \frac{5}{9}y_2 + \frac{13}{18}(1 - y_2) = \frac{13}{18} - \frac{y_2}{6} \geq \frac{13}{18} - \frac{5}{36} = \frac{7}{12},$$

as required. If $y_2 \geq 5/6$, then select r_{45} anchored at p_2 and a zero-area rectangle anchored at p_1 :

$$a_{45} = x_2y_2 \geq \frac{13}{18} \cdot \frac{5}{6} = \frac{65}{108} > \frac{63}{108} = \frac{7}{12},$$

as required.

Case 6: $x_1 \geq 4/9$. Select r_{14} and the largest of the 4 empty rectangles anchored at p_2 ; the total area is

$$A \geq x_1 + \frac{1 - x_1}{4} = \frac{1 + 3x_1}{4} \geq \frac{7}{12},$$

as required.

This completes the case analysis and thereby the proof of the lower bound. To see that this lower bound is the best possible, put $x_1 = 1/3$ and $x_2 = y_2 = 1/2$ and verify that no system of anchored rectangles covers an area larger than $7/12$. \square

B Proof of Theorem 2 (continued)

The same general idea applies to all cases (as well as the notation in the figures): two adjacent axis-aligned rectangles are specified, one empty of points and the other containing points, that allows the selection of an empty square of a suitable size anchored at an extreme point of the nonempty rectangle. The relevant subcases are illustrated in the figures (nonempty squares or subsquares are shaded) and the compact arguments are summarized in the tables. Any missing case is either similar or obtained by reflection of a present case with respect to a line. In each case:

- we consider a quadtree subdivision up to a certain level;
- in nonempty squares in the subdivision (shaded in the figures), some anchored square packing covers c times the area of corresponding squares by induction;
- we find new anchored squares of area $1/4$, $1/16$, $1/64$, or other, (in bold lines), which partially cover some empty and nonempty squares of the subdivision;
- the new anchored squares supersede the previous packing (i.e., obtained by induction) in the nonempty squares of the subdivision that they intersect.

We repeatedly make use of the following easy fact:

Observation 3. *Let $R_1, R_2 \subseteq U$ be two axis-aligned interior-disjoint rectangles of sizes $a_1 \times b$ and $a_2 \times b$, respectively, where $a_2 \geq b/2$, sharing a common edge of length b . If $R_1 \cap P \neq \emptyset$ and $\text{int}(R_2) \cap P = \emptyset$, then $R_1 \cup R_2$ contains an anchored empty square whose area is at least $b^2/4$.*

Proof. Assume that R_1 lies left of R_2 and let $p \in P$ be the rightmost point in R_1 . If p lies in the lower half-rectangle of R_1 then the square of side $b/2$ whose lower-left anchor is p is empty and has area $b^2/4$. Similarly, if p lies in the higher half-rectangle of R_1 then the square of side $b/2$ whose upper-left anchor is p is empty and has area $b^2/4$. \square

To specify anchor points, we use the following notation. Let $\emptyset \neq Q \subset P$. Let $x^-(Q)$ and $x^+(Q)$ denote the leftmost and rightmost points of Q , respectively (these points coincide if $|Q| = 1$). Let $y^-(Q)$ and $y^+(Q)$ denote the lowest and highest points of Q , respectively (these points coincide if $|Q| = 1$).

We next proceed with the details. In Cases 1 and 2, subdivide U_i , $i = 1, \dots, 4$ (in Case 1, U_2 , only) into four congruent squares U_{ij} , $j = 1, \dots, 4$, labeled counterclockwise around the center of U_i according to the subquadrant containing the square. Then partition P_i , $i = 1, \dots, 4$ into four subsets P_{ij} , $j = 1, 2, 3, 4$, $P_{ij} \subset U_i$ (ties are broken as specified earlier). We distinguish several subcases depending on the number of empty sets P_{ij} , $i = 2$, $j = 1, 2, 3, 4$, in the second level of the quadtree subdivision.

Refined analysis of Case 1 ($P_1 = \emptyset$, only). Let $\lambda_2 \in \{0, 1, 2, 3\}$ denote the number of empty P_{2j} . The relevant subcases are illustrated in Fig. 15 and summarized in Table 2. For example, in the first two rows in the table there are two inductive instances of area $1/4$ and two inductive instances of area $1/16$, contributing $2c/4$ and $2c/16$, respectively, toward the total covered area.

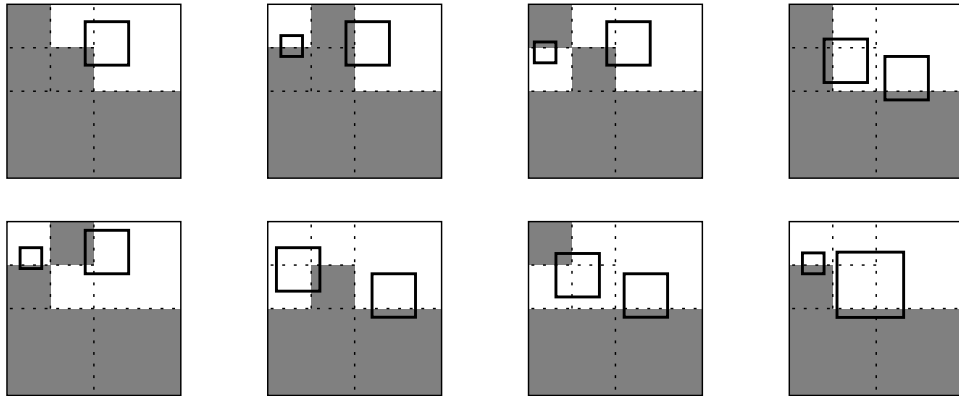


Figure 15: Case 1: the number of empty P_{2j} is in $\{0, 1, 2, 3\}$; rows: 1, 2; left to right: a, b, c, d .

Figure	λ_2	Anchor(s)	Inductive/direct proof	Resulting c
	0	$x^+(P_{21} \cup P_{24})$	$c = \frac{2c}{4} + \frac{2c}{16} + \frac{1}{16}$	$\frac{1}{6} > \frac{5}{32}$
15 (1a)	1	$x^+(P_{21} \cup P_{24})$	$c = \frac{2c}{4} + \frac{2c}{16} + \frac{1}{16}$	$\frac{1}{6} > \frac{5}{32}$
15 (1b,1c)	2	$y^+(P_{23}), x^+(P_{21} \cup P_{24})$	$c = \frac{2c}{4} + \frac{1}{16} + \frac{1}{64}$	$\frac{5}{32}$
15 (1d)	2	$x^+(P_{22} \cup P_{23}), y^+(P_4)$	$c = \frac{c}{4} + \frac{2}{16}$	$\frac{1}{6} > \frac{5}{32}$
15 (2a)	2	$y^+(P_{23}), x^+(P_{21} \cup P_{24})$	$c = \frac{2c}{4} + \frac{1}{16} + \frac{1}{64}$	$\frac{5}{32}$
15 (2b,2c)	3	$x^-(P_{21} \cup P_{24}), y^+(P_4)$	$c = \frac{c}{4} + \frac{2}{16}$	$\frac{1}{6} > \frac{5}{32}$
15 (2d)	3	$y^+(P_{23}), y^+(P_{31} \cup P_{34} \cup P_4)$	$c = \frac{9}{64} + \frac{1}{64}$	$\frac{5}{32}$

Table 2: Inductive/direct proofs for the subcases of Case 1.

Refined analysis of Case 2. If the two nonempty squares are adjacent, say $P_2, P_3 \neq \emptyset$, then one of the two squares of side $1/2$, whose lower-left or upper-left anchor is the rightmost point in $P_2 \cup P_3$ is empty and contained in U ; its area is $1/4 > 5/32$, as required. Assume henceforth that the two nonempty squares lie in opposite quadrants, e.g., $P_2, P_4 \neq \emptyset$. For $i = 2, 4$, let λ_i denote the number of empty sets P_{ij} ; by symmetry, we can assume that $\lambda_2 \leq \lambda_4$; thus $0 \leq \lambda_2 \leq \lambda_4 \leq 3$. We further distinguish Cases 2a, 2b, 2c, 2d, as determined by the different values of λ_2 .

Case 2a: $\lambda_2 = 0$. The relevant subcases are illustrated in Fig. 16 and summarized in Table 3. Fig. 16 (3b) and Fig. 16 (3c) correspond to $|P_{42}| \geq 2$ and $|P_{42}| = 1$, respectively.

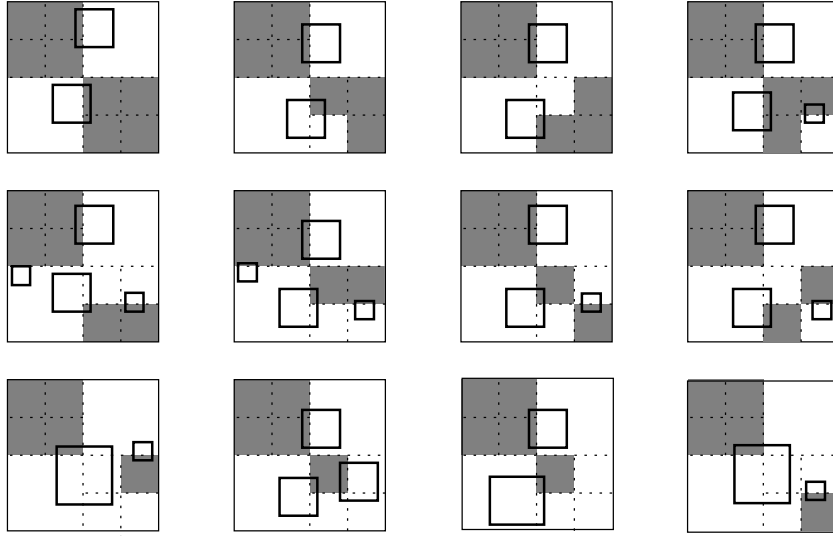


Figure 16: Case 2a: the number of empty P_{2j} is 0; rows: 1, 2, 3; left to right: a, b, c, d .

Case 2b: $\lambda_2 = 1$. The relevant subcases for $\lambda_4 = 1, 2, 3$ are illustrated in Figures 17, 18, 19, respectively; the proofs are summarized in Table 4.

For example: if $\lambda_4 = 3$, so that $P_{42} \neq \emptyset$ as in Fig. 19 (1b,2b,3b), we proceed depending on whether $|P_{42}| \geq 2$ or $|P_{42}| = 1$. If $|P_{42}| \geq 2$ we proceed as in Fig. 19 (1b); if $|P_{42}| = 1$ and the corresponding point, say p , lies in $U_{422} \cup U_{423}$, then p is set as a left anchor and one can get an empty covered area at least $9/64 + 1/64 = 5/32$; if $|P_{42}| = 1$ and the corresponding point, say p , lies in $U_{421} \cup U_{424}$, then p is set as an upper-left anchor and the lowest point in P_{24} is set as an upper anchor as shown in Fig. 19 (2b,3b), with an empty covered area at least $25/256 + 1/16 = 41/256 > 5/32$.

Case 2c: $\lambda_2 = 2$. The relevant subcases are illustrated in Fig. 20 and summarized in Table 5.

Figure	λ_2	λ_4	Anchor(s)	Inductive/direct proof	Resulting c
16 (1a)	0	0	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43})$	$c = \frac{4c}{16} + \frac{2}{16}$	$\frac{1}{6} > \frac{5}{32}$
16 (1b,1c)	0	1	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43})$	$c = \frac{4c}{16} + \frac{2}{16}$	$\frac{1}{6} > \frac{5}{32}$
16 (1d)	0	1	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43}), \dots$	$c = \frac{2c}{16} + \frac{2}{16} + \frac{1}{64}$	$\frac{9}{56} > \frac{5}{32}$
16 (2a,2b)	0	2	$x^+(P_{21} \cup P_{24}), y^-(P_{23}), \dots$	$c = \frac{2}{16} + \frac{2}{64}$	$\frac{5}{32}$
16 (2c,2d)	0	2	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43}), \dots$	$c = \frac{2c}{16} + \frac{2}{16} + \frac{1}{64}$	$\frac{9}{56} > \frac{5}{32}$
16 (3a)	0	3	$y^-(P_{23} \cup P_{24}), y^+(P_{41})$	$c = \frac{9}{64} + \frac{1}{64}$	$\frac{5}{32}$
16 (3b)	0	3	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43}), \dots$	$c = \frac{3}{16}$	$\frac{3}{16} > \frac{5}{32}$
16 (3c)	0	3	$x^+(P_{21} \cup P_{24}), x^-(P_{42} \cup P_{43})$	$c \geq \frac{1}{16} + \frac{25}{256}$	$\frac{41}{256} > \frac{5}{32}$
16 (3d)	0	3	$y^-(P_{23} \cup P_{24}), y^+(P_{44})$	$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$

Table 3: Inductive/direct proofs for Case 2a.

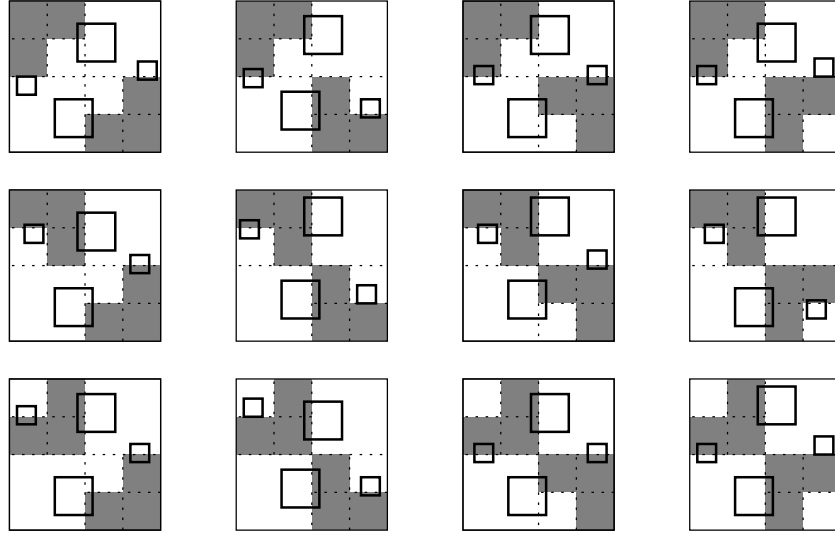


Figure 17: Case 2b with one empty P_{2j} and one empty P_{4j} ; rows: 1, 2, 3; left to right: a, b, c, d .

Figure	λ_2	λ_4	Anchor(s)	Direct proof	Resulting c
17 (1a,1b,1c,1d)	1	1	$x^+(P_{21} \cup P_{24}), y^-(P_{23}), \dots$	$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
17 (2a,2b,2c,2d)	1	1	$x^+(P_{21} \cup P_{24}), y^-(P_{22}), \dots$	$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
17 (3a,3b,3c,3d)	1	1	$x^+(P_{21} \cup P_{24}), y^+(P_{23}), \dots$	$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
18 (1a through 4d)	1	2		$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
19 (1a,1c)	1	3		$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$
19 (1b)	1	3		$c \geq \frac{25}{256} + \frac{1}{16}$	$\geq \frac{5}{32}$
19 (1d)	1	3		$c \geq \frac{1}{4}$	$\geq \frac{1}{4} > \frac{5}{32}$
19 (2a,2c,2d)	1	3		$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$
19 (2b)	1	3		$c \geq \frac{25}{256} + \frac{1}{16}$	$\geq \frac{5}{32}$
19 (3a,3c,3d)	1	3		$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$
19 (3b)	1	3		$c \geq \frac{25}{256} + \frac{1}{16}$	$\geq \frac{5}{32}$

Table 4: Direct proofs for Case 2b.

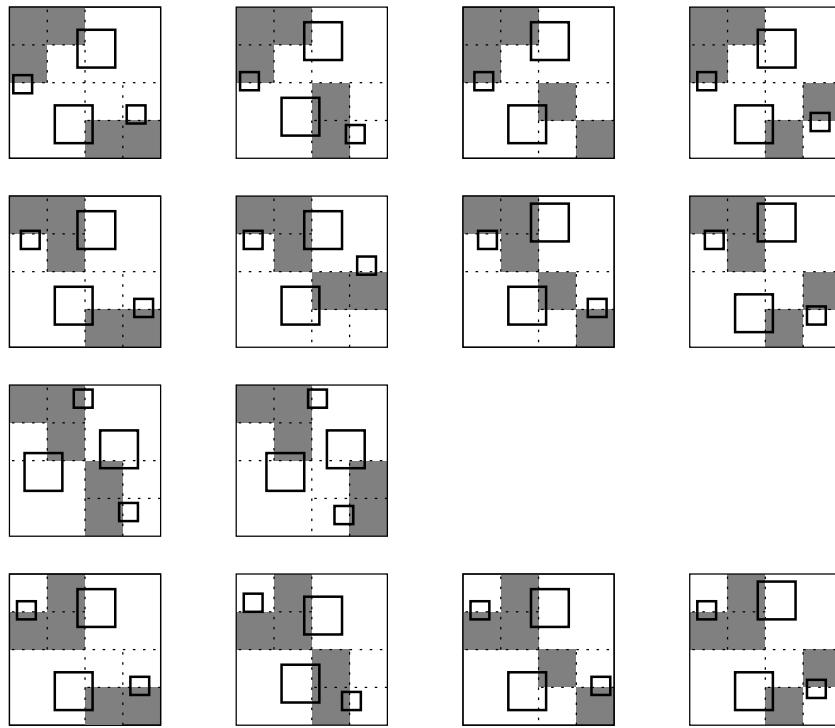


Figure 18: Case 2b with one empty P_{2j} and two empty P_{4j} ; rows: 1, 2, 3, 4; left to right: a, b, c, d .

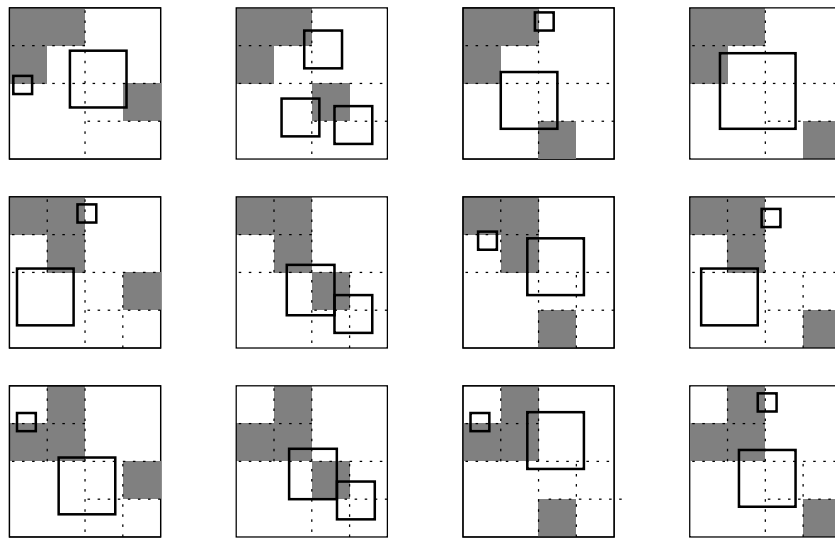


Figure 19: Case 2b with one empty P_{2j} and three empty P_{4j} ; rows: 1, 2, 3; left to right: a, b, c, d .

Assume first that $\lambda_4 = 2$. If P_{22} and P_{24} are nonempty, as in Fig. 20 (1a,1b), we proceed according to whether the two nonempty P_{4j} are in adjacent vertical strips of width $1/4$ or in the same vertical strip of width $1/4$: in the former case, we select empty squares as in Fig. 20 (1a) while in the latter case, we select empty squares as in Fig. 20 (1b). If P_{21} and P_{23} are nonempty, as in Fig. 20 (1c,1d), we proceed analogously. If P_{22} and P_{23} are nonempty, as in Fig. 20 (2a,2b), it suffices to consider the cases when the two nonempty squares P_{4j} are in the same strip of width $1/4$ (the complementary cases have been covered in row 1); then the empty covered area is at least $9/64 + 1/64 = 5/32$, or at least $3/16 > 5/32$, as required. If P_{21} and P_{24} are nonempty, as in Fig. 20 (2c,2d), it again suffices to consider the cases when the two nonempty squares P_{4j} are in the same strip of width $1/4$, and then the empty covered area is at least $3/16 > 5/32$, as required.

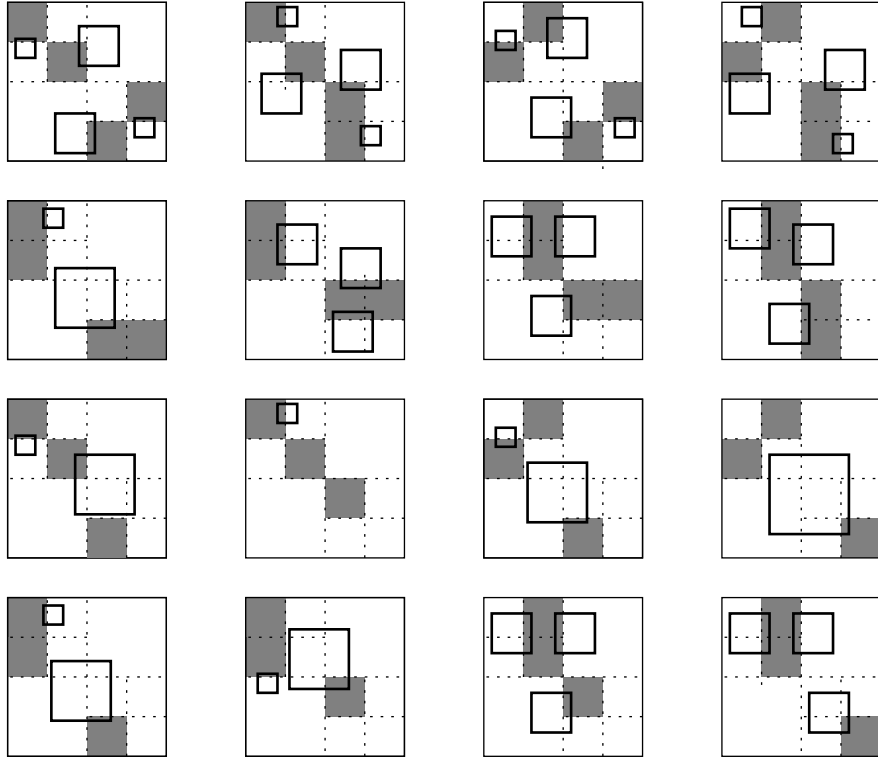


Figure 20: Case 2c: two empty P_{2j} and two or three empty P_{4j} ; rows: 1, 2, 3, 4; left to right: a, b, c, d .

Assume now that $\lambda_4 = 3$. If P_{22} and P_{24} are nonempty, as in Fig. 20 (3a,3b), we proceed as follows¹: if P_{41} , P_{43} or P_{44} are nonempty, the empty covered area is at least $9/64 + 1/64 = 5/32$; if P_{42} is nonempty, consider $|P_{22}|$, $|P_{24}|$, and $|P_{42}|$. If $|P_{24}| \geq 2$ or $|P_{42}| \geq 2$, select two empty squares of area $1/16$ anchored at points of the respective “large” set, and another empty square of area $1/16$ anchored at the point of the other set (among P_{24} and P_{42}); we thereby have a total empty area $3/16 > 5/32$, as required. If $|P_{24}| = |P_{42}| = 1$, the empty covered area is at least $9/64 + 1/64 = 5/32$, unless the point in P_{24} , say p , lies in U_{242} and the point in P_{42} , say q , lies in U_{424} ; then an empty square whose upper-left anchor is p and side $x(q) - x(p)$ and an empty square whose lower-left anchor is q and side $1 - x(q)$ cover an area at least $2 \left(\frac{5}{16}\right)^2 = \frac{25}{128} > \frac{5}{32}$, as required. If P_{21} and P_{23} are nonempty, as in Fig. 20 (3c,3d), the empty covered area is at least $9/64 + 1/64 = 5/32$, or at least $1/4$, as required. If P_{22} and P_{23} are nonempty, as in Fig. 20 (4a,4b),

¹Observe that our quadtree decomposition analysis “matches” the lower bound construction in Proposition 2 in the sense that the hardest case to deal with in our proof is the one in which the lower bound construction fits in.

there are two possibilities: P_{4j} is nonempty where $j \in \{1, 2\}$, or where $j \in \{3, 4\}$; each case is dealt with easily. If P_{21} and P_{24} are nonempty, as in Fig. 20 (4c,4d), each case is dealt with easily.

Case 2d: $\lambda_2 = 3$. Since $\lambda_2 \leq \lambda_4$, we have $\lambda_4 = 3$. The relevant subcases are illustrated in Fig. 21 and summarized in Table 5. If P_{21} or P_{22} or P_{23} are nonempty, as in Fig. 21 (1a,1b), the empty covered area is at least $9/64 + 1/64 = 5/32$, as required. If P_{24} is nonempty, as in Fig. 21 (1c), at least one of the nonempty sets has at least 2 points, and so the empty covered area is at least $3/16 > 5/32$, as required.

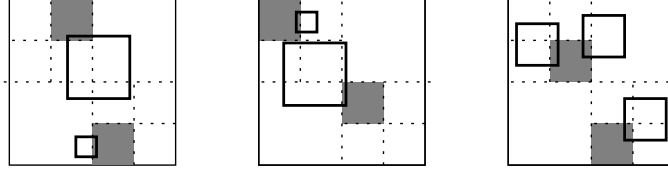


Figure 21: Case 2d: three empty P_{2j} and three empty P_{4j} ; left to right: a, b, c .

Figure	λ_2	λ_4	Direct proof	Resulting c
20 (1a,1b)	2	2	$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
20 (1c,1d)	2	2	$c \geq \frac{2}{16} + \frac{2}{64}$	$\geq \frac{5}{32}$
20 (2a,2b)	2	2	$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$
20 (2c,2d)	2	2	$c \geq \frac{3}{16}$	$\geq \frac{3}{16} > \frac{5}{32}$
20 (3c,4a,4b)	2	3	$c \geq \frac{9}{64} + \frac{1}{64} = \frac{5}{32}$	$\geq \frac{5}{32}$
20 (3d)	2	3	$c \geq \frac{1}{4}$	$\geq \frac{1}{4} > \frac{5}{32}$
20 (4c,4d)	2	3	$c \geq \frac{3}{16}$	$\geq \frac{3}{16} > \frac{5}{32}$
21 (a,b)	3	3	$c \geq \frac{9}{64} + \frac{1}{64}$	$\geq \frac{5}{32}$
21 (c)	3	3	$c \geq \frac{3}{16}$	$\geq \frac{3}{16} > \frac{5}{32}$

Table 5: Direct proofs for Case 2c and Case 2d.

This completes the case analysis and thereby the proof of Theorem 2. □