

2.1 Sets

Definition: A *set* S is an unordered collection of distinct objects, called *elements* of the set. We let $a \in S$ to mean that a is an element of set S . (Reminder: the order of the elements as well as their multiplicities do not matter!)

For example, the set of vowels is $\{a, e, i, o, u\}$. The set of integers is $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. The set of rational numbers is denoted by \mathbf{Q} , while the set of real number is \mathbf{R} . The empty set is denoted by $\emptyset = \{\}$.

To describe a set, we can either (i) list all the elements of the set or (ii) use set builder notation which describes what should be in the set.

Definition: Two sets A and B are *equal* if and only if they have exactly the same elements. Their equality is expressed as $A = B$.

Definition: A set A is a *subset* of a set B if and only if every element of A is an element of B . We use the notation $A \subseteq B$. We also say that A is a *proper subset* of B when A is a subset of B but $A \neq B$. In this case, the notation is $A \subset B$.

What is the subset relationships between $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$?

What about \emptyset and any set S ?

Definition: The *cardinality* of set S , $|S|$, is the number of elements in S .

If $|S| = n$, where n is a non-negative integer, we say that S is a *finite* set. Otherwise, S is an *infinite* set.

Power Sets

Definition: The *power set* of S , $\mathcal{P}(S)$, is the set containing all the subsets of S .

Example: What is the power set of $S = \{a, b, c\}$?

How about the power set of \emptyset ?

If $|S| = n$, what is $|\mathcal{P}(S)|$? That is, how many subsets does S have?

Cartesian Products

Definition: Let A and B be sets. The *Cartesian product* of A and B is the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$. We denote the set as $A \times B = \{(a, b) | a \in A, b \in B\}$.

Example: Let $A = \{\text{wheat, white}\}$, $B = \{\text{ham, roast beef, tuna}\}$. What is $A \times B$?
 $B \times A$?

Some notes:

Definition: The *Cartesian product* of sets A_1, A_2, \dots, A_n is the set of all ordered n -tuples (a_1, \dots, a_n) where $a_i \in A_i$ for $i = 1, \dots, n$. That is,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) : a_i \in A_i, i = 1, \dots, n\}.$$

2.2 Set Operations

Like propositions, there are various ways of creating new sets from old ones. Let A and B be sets, and U be the universal set where the elements of A and B reside. Here are some important set operations:

1. The *union* of A and B .

2. The *intersection* of A and B .

If $A \cap B = \emptyset$, A and B are said to be *disjoint sets*.

3. The *difference* of A and B .

4. The *complement* of A .

Set Identities

As in the case of propositions, we often want to show that two sets are equal. Here's a basic technique for proving that $A = B$: for every element x , $x \in A$ if and only if $x \in B$. That is, show that $A \subseteq B$ and $B \subseteq A$.

Example: Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Example: Show that $(B - A) \cup (C - A) = (B \cup C) - A$.

Example: Show that $(A - C) \cap (C - B) = \emptyset$.

Another method of course is to use set identities.

Example: Using set identities, show that $(B - A) \cup (C - A) = (B \cup C) - A$.

Example: Using set identities, show that $\overline{A \cap (B \cup C)} = (\overline{B} \cap \overline{C}) \cup \overline{A}$.

Generalized unions and intersections

Definition: Let A_1, \dots, A_n be sets.

1. $\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$ is the set that contains all the elements that are members of *at least one* set in the collection.
2. $\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$ is the set that contains all the elements that are members of *every* set in the collection.

Exercise: What is the most general statement you can make about sets A and B if you know that

1. $A \cup B = A$?
2. $A \cap B = A$?
3. $A - B = A$?
4. $A \cap B = B \cap A$?
5. $A - B = B - A$?
6. $\bar{A} = \bar{B}$?
7. $\bar{A} \subseteq \bar{B}$?

More on set cardinalities

Theorem 1 *Suppose A and B are sets. Then*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example. How many integers are there between 1 and 100 (inclusive) that are even or divisible by 5?