Name: ______________________________ May 14, 2009

CS 317 Final Exam

Instructions: There are eight questions in this exam, each with multiple subquestions. Make sure that you read through them carefully, providing as thorough an answer as you can.

1. (5 pts.) Let $P(x), Q(x), R(x)$ be the statements "x is a clear explanation," "x is satisfactory," "x is an excuse," respectively. Suppose that the domain for x consists of all the English text. Express each of these statements using quantifiers, logical connectives and $P(x), Q(x), R(x)$.

   a. All clear explanations are satisfactory.

   b. Some excuses are unsatisfactory.

   c. Some excuses are not clear explanations.

   d. Finally, please determine if statement c follows from statements a and b. If yes, please show why this is the case using rules of inference; if not, please provide an explanation.

2. (5 pts.) Consider the following statement: “If x is an even number and y is an even number then $x + y$ is an even number.”

   a. Please state the contrapositive of this statement.

   b. Please state the converse of this statement.

   c. Please prove the statement.

   d. Is its contrapositive true? Why or why not?

   e. How about its converse? Why or why not?

3. (5 pts.) Let A and B be sets. For each statement below, please write down the most general statements you can make about A and B. Make sure you justify your answer.

   a. $A \cup B = A$?

   b. $A \cap B = A$?

   c. $A \cup B = A \cap B$?

   d. $A - B = A$?

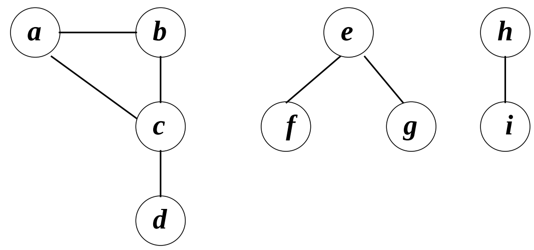
   e. $A - B = B - A$?

4. (8 pts.) Let $G = (V, E)$ be a simple, undirected graph. Define a relation R on the vertices of G as follows: for two nodes u and v, $(u, v) \in R$ if and only if there is a path from u to v in G.

   a. Determine if R is reflexive, symmetric, anti-symmetric, transitive or total. For each one, please provide an explanation as to why your answer is yes or no.

   b. Why is R an equivalence relation?

   c. Suppose G is the graph below. What are the equivalence classes of R?



5. (5 pts.) *Let's give this another try.* How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$ where $x_1, x_2, x_3,$ and x_4 are nonnegative integers with

a. $x_3 = 2$ and $x_4 = 3$?

b. $x_3 + x_4 = 5$?

c. $x_1 \neq 5$?

d. $x_1 < 6$?

e. $x_1 < 6$ and $x_3 > 5$?

6. (5 pts.) Five boys and five girls attended a concert. Since the concert hall was crowded, they had to split up. Five of them sat together on row one, the other five sat together on row two.

   a. An arrangement in this situation can be described as a sequence $(x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5)$ where $(x_1, x_2, x_3, x_4, x_5)$ are the names of the kids that sat on the first row from left to right, while $(y_1, y_2, y_3, y_4, y_5)$ are the names of the kids that sat on the second row from left to right. How many arrangements are possible?

   Suppose all arrangements are equally likely. Please compute the *probabilities* of the following events:

   b. All boys sat together (and so all girls sat together too)?

   c. Two girls, Amy and Beth, sat next to each other in the first row?

   d. All boys sat together *and* Amy and Beth are next to each other in the first row?

   e. Amy and Beth are next to each other in the first row given that the five boys sat together?

7. (5 pts.) In class, we briefly discussed the notion of a *complete tripartite graph* $K_{l,m,n}$. It is a graph whose vertex set can be partitioned into three parts V_1, V_2, V_3 so that $|V_1| = l, |V_2| = m$ and $|V_3| = n$. Additionally, each vertex in one partite set is adjacent to every vertex in the other two partities sets. Thus, it is just like a complete bipartite graph – except that instead of two partite sets, we now have three partite sets.

   a. Please draw $K_{2,2,3}$.

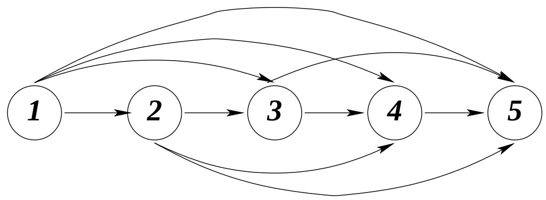
   b. For *general* values of l, m, and n, how many vertices are in $K_{l,m,n}$?

   c. How many edges are in $K_{l,m,n}$?

   d. For what values of l, m and n does $K_{l,m,n}$ have an Euler circuit?

   e. For what values of l, m and n does $K_{l,m,n}$ have an Euler trail?

8. (5 pts.) Consider a directed graph $D(n)$ whose vertex set V consists of the integers $1, 2, \ldots, n$, and whose edge set E consists of ordered pairs of integers (i, j) whenever $i < j$. In other words, there is a directed edge from i to j if and only if $i < j$. For example, the figure below shows $D(5)$.



Our goal is to count the number of paths from 1 to n in $D(n)$.

a. Warm-up! In $D(n)$, a path from 1 to n is described by a sequence of vertices $v_1, v_2, \ldots, v_k$ where $v_1 = 1$, $v_i < v_{i+1}$ for $i = 1, \ldots, k-1$, and $v_k = n$. For $D(5)$ above, please list all the paths from 1 to 5. (There should be 8 of them.)

What makes two paths from 1 to n different is the "middle" part of their sequences. For example, in $D(5)$ above, the path $1, 2, 4, 5$ is different from $1, 3, 5$ because the middle part of the first path is $2, 4$ whereas the middle part of the second path is 3. To accomplish our goal then, we need to determine how many different "middle" parts a path from 1 to n can have. We will do this by defining a bijective function $f : A \to B$ where A is the set of paths from 1 to n in $D(n)$, and B is the power set of $\{2, 3, \ldots, n-1\}$. (Recall that the power set of $\{2, 3, \ldots, n-1\}$ consists of all subsets of $\{2, 3, \ldots, n-1\}$).

b. Suppose $P = v_1, v_2, \ldots, v_k$ is a 1-to-n path in $D(n)$. Let $f(P) = \{v_2, \ldots, v_{k-1}\}$. For example, for $D(5)$, $f(1, 2, 4, 5) = \{2, 4\}$. Please explain why f is one-to-one.

c. Next, explain why f is onto.

d. From b and c, we conclude that f is a bijection. What then does this imply about the number of 1-to-n paths in $D(n)$? (Note: your answer here should be a generalization of your answer in (a).)