Name: ______________________________ November 30, 2009

CS 317 Exam II

Instructions: There are six problems in this exam. Some of them consist of several subproblems. Make sure you look through them carefully. Please write legibly and justify all your answers.

1. Our goal is to use mathematical induction to show that the following equation is true whenever n is a positive integer:

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = (n-1)2^n + 1.$$

   a. What is $P(n)$?

   b. Show that $P(1)$ is true.

   c. What is the inductive step?

   d. Prove the inductive step.

   Answers:

   a. $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = (n-1)2^n + 1$.

   b. LHS is equal to $1 \times 2^0 = 1$ while RHS is equal to $0 \times 2^1 + 1 = 1$. Hence $P(1)$ is true.

   c. That $P(k)$ implies $P(k+1)$ for $k = 1, 2, \ldots$.

   d. Assume $P(k)$ is true. Now, consider the LHS of $P(k+1)$:

$$\begin{aligned} 1 \times 2^0 + 2 \times 2^1 + \ldots + k \times 2^{k-1} + (k+1) \times 2^k &= (k-1)2^k + 1 + (k+1) \times 2^k \\ &= 2k \times 2^k + 1 \\ &= k \times 2^{k+1} + 1. \end{aligned}$$

   That is, the LHS of $P(k+1)$ is equal to its RHS. The first equality is true because of $P(k)$; the rest is just algebra. Hence, by induction $P(n)$ is true for positive integers n.

2. Let the set S consist of the numbers 1 to 100.

a. How many ways can 10 numbers from S be arranged on a line?

b. What if the 10 numbers have to be in increasing order?

c. What if the numbers alternate between being even and odd? (e.g., $5, 40, 7, 38, 9, 36, 11, 34, 13, 32$)

d. What if the numbers 50, 60 and 70 have to be among the ten numbers on the line? (e.g., $87, 5, 60, 11, 15, 50, 21, 22, 23, 70$)

e. What if the numbers 50, 60 and 70 are among the ten numbers and 50 is to the left of 60 and 60 is to the left of 70 on the line? (e.g., $87, 5, 50, 11, 15, 60, 21, 22, 23, 70$)

Answers:

a. $P(100, 10) = 100 \times 99 \times \ldots \times 91$.

b. $C(100, 10)$. To construct ten numbers from S in increasing order, first choose the ten numbers from S and then arrange them in increasing order. There's only one way to do the latter step.

c. $2 \times P(50, 5) \times P(50, 5)$. There are two patterns for odd and even numbers alternating on a line. Once the pattern is chosen, arrange the five odd numbers. Then arrange the five even numbers.

d. $10 \times 9 \times 8 \times P(97, 7) = P(10, 3) \times P(97, 7)$. Pick a spot for 50 then 60 then 70. Then place seven other numbers in the remaining spots.

e. $C(10, 3) \times P(97, 7)$. Choose three spots from the ten spots. Place 50 in the leftmost one, 60 in the middle, 70 in the rightmost one. Then place seven other numbers in the remaining spots.

3. Suppose that a bagel shop sells eight different kinds of bagels.

   a. How many ways are there to choose a dozen bagels?

   Suppose the shop packs a dozen bagels for you at random. What is the probability that it contains

   b. at least two egg bagels?

   c. no more than six salty bagels?

   d. at least two egg bagels *and* no more than six salty bagels?

   e. at least two egg bagels *or* no more than six salty bagels?

   Answers: For this problem, the bagels are the indistinguishable balls and the 8 varieties are the distinct boxes.

   a. $C(12+8-1,12) = C(19,12)$

   For (b) to (e), let E_1 be the event that a dozen bagels chosen at random contain at least two egg bagels. Let E_2 be the event that a dozen bagels contain no more than six salty bagels. Thus, the complement of E_2, $\overline{E_2}$, is the event that a dozen bagels contain at least seven salty bagels.

   b. $Prob(E_1) = C(10+8-1,10)/C(19,12)$.

   c. $Prob(E_2) = 1 - Prob(\overline{E_2}) = 1 - C(5+8-1,5)/C(19,12)$.

   d. $Prob(E_1 \cap E_2) = Prob(E_1) - Prob(E_1 \cap \overline{E_2}) = C(10+8-1,10)/C(19,12) - C(3+8-1,3)/C(19,12)$.

   e. $Prob(E_1 \cup E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \cap E_2)$. So simply substitute the answers from (b), (c) and (d) to get the answer.

4. A researcher wanted to understand how married couples influence each other's TV-watching habits. After doing an extensive survey, she noted that the probability that a married man watches a certain television show is 0.4 while the probability that a married woman watches the show is 0.5. The probability that a married man watches the show, *given that his wife does*, is 0.7. (Note this is a conditional probability.)

   Before we proceed, let us give names to the events described above. Let M and W be the events that a married man and a married woman watches the show respectively. According to what's given above, $Prob(M) = 0.4$, $Prob(W) = 0.5$ and $Prob(M|W) = 0.7$.

   Find the probability that

   a. a man *and* his wife watches the show? (Warning: M and W are *not* independent events!)

   b. a wife watches the show *given that her husband does*?

   c. at least one person of a married couple (a man or his wife) watches the show?

   Answers:

   a. Recall that $Prob(M|W) = \frac{Prob(M \cap W)}{Prob(W)}$. Thus, $Prob(M \cap W) = Prob(M|W) \times Prob(W) = 0.7 \times 0.5 = 0.35$.

   b. $Prob(W|M) = \frac{Prob(M \cap W)}{Prob(M)} = \frac{0.35}{0.4}$

   c. $Prob(M \cup W) = Prob(M) + Prob(W) - Prob(M \cap W) = 0.4 + 0.5 - 0.35 = 0.55$.

5. A couple is planning to have a family. Let us assume that the probability of having a girl is 0.48 and a boy is 0.52, and that the births of this couple's children are pairwise independent. They want to have at least one girl and at least one boy. At the same time, they know that raising more than 5 kids will be very difficult. So here's what they plan to do: they'll keep trying to have children until they have at least one girl and at least one boy or until they have five kids. Our goal is to determine the expected number of children this couple will have.

   Let $X(s)$ be equal to the number of children with outcome s. Notice that $X(s)$ is at least 2 and at most 5; e.g., $X(GGGB) = 4$.

   a. For $i = 2, 3, 4, 5$, list the outcomes in the event $(X = i)$.

   b. For $i = 2, 3, 4, 5$, what is $P(X = i)$?

   c. What is $E[X]$?

   Answers:

   a. $(X = 2) = \{GB, BG\}$

   $(X = 3) = \{GGB, BBG\}$

   $(X = 4) = \{GGGB, BBBG\}$

   $(X = 5) = \{GGGGB, BBBBG, GGGGG, BBBBB\}$

   b. $P(X = 2) = 2 \times 0.48 \times 0.52$.

   $P(X = 3) = 0.48^2 \times 0.52 + 0.52^2 \times 0.48$.

   $P(X = 4) = 0.48^3 \times 0.52 + 0.52^3 \times 0.48$.

   $P(X = 5) = 0.48^4 \times 0.52 + 0.52^4 \times 0.48 + 0.48^5 + 0.52^5$.

   c. $E[X] = 2 \times 2 \times 0.48 \times 0.52 + 3 \times (0.48^2 \times 0.52 + 0.52^2 \times 0.48) + 4 \times (0.48^3 \times 0.52 + 0.52^3 \times 0.48) + 5 \times (0.48^4 \times 0.52 + 0.52^4 \times 0.48 + 0.48^5 + 0.52^5)$.

6. You play a game in which you roll a six-sided die and you win (in dollars) the square of the number on the die. Let X be the random variable that denotes the payoff you receive when the game is played. For example, $X(s) = s^2$ for $s = 1, 2, \ldots, 6$.

   a. Solve for $E[X]$, the average money you would expect to receive per play on this game.

   b. Solve for $Var[X]$. A pair of dice is rolled in a remote location which is observed by some honest person.

   Answers:

   a. $E[X] = \sum_{s \in S} s^2 Prob(s) = \sum_{s \in S} s^2 \frac{1}{6} = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{1}{6}(91)$.

   b. $Var[X] = E[X^2] - E[X]^2$. Thus, we just have to compute $E[X^2]$ and then use the answer from (a).

   $E[X^2] = \frac{1}{6}(1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4)$.