5.1 Discrete Probability

The origin of probability dates back to the 17th century when Blaise Pascal first analyzed some gambling games. In the 18th century, Pierre-Simon Laplace defined the *probability of an event* as

$$\frac{\text{number of outcomes that belong to the event}}{\text{number of possible outcomes}}.$$

For example, if you throw a die (the singular form of dice), the probability that the result is an even number is 1/2.

Definition In an experiment, the *sample space* is the set of all possible outcomes. An *event* is a subset of the sample space.

Example Suppose two dice are thrown. What is the sample space? What does the event where the sum of the numbers on the faces of the dice equals four consist of?

Example In a Pick-6 lottery, six numbers are chosen from the numbers 1 to 40 without repetition. What is the sample space?

Definition Suppose the outcomes of an experiment are *equally likely*. Let S be the sample space and E be an event. The probability of E is

$$p(E) = \frac{|E|}{|S|}.$$

Exercises

1. Suppose two dice are thrown. What is the probability that the sum of the two numbers equal four?

2. Suppose the Pick-6 lottery was administered using a fair mechanism so that all outcomes are equally likely. You bought ten tickets no two of which contain the same set of numbers. What is the probability that you will win the lottery?

3. In a drawer, there are six distinct pairs of socks whose right and left sock can also be distinguished from each other. If we pick two socks at random, what is the probability that the socks match?

4. Suppose that a woman is equally likely to give birth to a girl and to a boy. If she plans to have ten children, what are the probabilities of these events?

   a. all boys?

   b. GGGGBGGGBG?

   c. 2 boys and 8 girls?

   d. 5 boys and 5 girls?

5. Suppose five cards were chosen uniformly at random from a deck of 52 cards. What is the probability of

   a. a full house?

   b. a straight?

5.2 Probability Theory

A more general approach. While Laplace's definition of probability captures the notion of the likelihood that an event will occur, it relies on the assumption that all outcomes are equally likely. In practice, this assumption does not always hold. For example, in a professional golf tournament where Tiger Woods is a participant, his chances of winning are greater than anyone else's. Here's a more general definition.

Definition Let S be the sample space of an experiment with a finite number of outcomes. We assign a *probability* $p(s)$ for each $s \in S$ so that the following are true:

(i) $0 \leq p(s) \leq 1$ for each $s \in S$ and
(ii) $\sum_{s \in S} p(s) = 1$.

The function $p : S \to [0, 1]$ is called a *probability distribution for* S.

Definition The *probability of an event* E is $p(E) = \sum_{s \in E} p(s)$.

- Previously, we assumed that the outcomes in S are equally likely. Hence, for each $s \in S$,

$$p(s) =$$

- But how is $p(s)$ derived in practice? This is a question of interest to statisticians. How likely will a pitcher throw a fastball? How likely will a football team on a third down situation run instead of throw? How reliable is a laptop made by Dell? Most of these questions can be answered by gathering data. In particular, suppose an experiment is conducted many times. (How many?) For each $s \in S$, set

$$p(s) = \frac{\text{number of times the outcome is } s}{\text{number of times experiment is repeated}}.$$

  For example, one can collect data from the previous year to estimate the probabilities of a woman giving birth to a boy versus a girl.

Example. Suppose a biased die is rolled so that except for 3, all other outcomes are equally likely, but 3 appears twice more often than each outcome. What is the probability that an even number appears?

Theorem 1 *Let p be a probability distribution over S. Let E and E' be events. The following are true:*

(i) $p(E) = 1 - p(\bar{E})$.

(ii) $p(E \cup E') = p(E) + p(E') - p(E \cap E')$.

Example. What is the probability that at least two people among a group of n people have the same birthday? Assume that it is equally likely for a person to be born on any day of the year.

Example. What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by 2 or 5?

Conditional Probabilities

Consider the following experiment. Suppose a fair coin is toss four times. What is the probability that at least one head appears?

Suppose we know that the first toss turned up tails. What is the probability of the same event?

Knowing the result of the first toss changed the probability of the event. This leads us to conditional probabilities.

Definition Let E and F be events with $p(F) > 0$. The *conditional probability of E given F* is

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

Example. Let D and A be the events that a regularly scheduled flight departs and arrives on time. Suppose that $p(D) = 0.83$ and $p(A) = 0.92$ and $p(D \cap A) = 0.78$. What is $p(A|D)$? $p(D|A)$?

Independence

There are times when knowing the occurrence of event F does not affect the probability of event E. That is, $p(E|F) = p(E)$. In other words,

$$\frac{p(E \cap F)}{p(F)} = p(E) \text{ or } p(E \cap F) = p(E)p(F).$$

This often occurs when an experiment is conducted multiple times. One usually assumes that the result of an earlier experiment does not affect those of later experiments – that is, the outcomes are independent.

Definition Events E and F are *independent* if $p(E \cap F) = p(E)p(F)$.

Example. Suppose a coin is biased so that $p(H) = 2/3$ and $p(T) = 1/3$. It is tossed five times. What is $P(HTTHT)$, assuming each coin toss is independent of others?

What is the probability that two heads are obtained?

Definition A *Bernoulli trial* is an experiment with only two possible outcomes: a "success" with probability p and a "failure" with probability $q = 1 - p$.

For example, coin tosses (H/T), lotteries (win/lose), giving birth (girl/boy), etc. are Bernoulli trials.

Theorem 2 *The probability of exactly k successes in n independent Bernoulli trials is*

Example. Suppose the probability of a woman giving birth to a girl and to a boy is 0.55 and 0.45 respectively. Assume that the births of the woman are independent from each other. What are the probability of these events?

a. all kids have the same gender?

b. GGGGBGGGBG?

c. 2 boys and 8 girls?

d. 5 boys and 5 girls?

Are they independent? In previous examples, we *assumed* that certain events are independent. In some situations, however, *the question* is whether events are indeed independent. For example, do previous results of a fair lottery influence its next result? That is, suppose in 1000 results of a Pick-6 lottery, the number 1 has only appeared once. Is it more likely to occur in the 1001st result than the other numbers?

To determine if E and F are independent, check if $p(E \cap F) = p(E)p(F)$.

Example. A bag contains 20 balls – 10 red and 10 blue. Two balls are drawn from the bag. Let A be the event that the first ball is red, and B be the event that the second ball is red. Are A and B independent?