1.3 Predicates and Quantifiers

We've talked about some statements before which are not propositions. For example, "Study hard!" or "What's next?". There were also statements like "$x > 3$, "$x+y = z$" which are *almost* propositions. Why almost?

Definition: A *propositional function* is a statement that contains variables. Once the values of the variables are specified, the function has a truth value.

Example: $P(x)$ = "$x > 3''$, $Q(x, y)$ = "x is the best player of team y", $R(x, y, z)$ = "$x + y = z''$.

To specify the value of the variables, we can certainly just say let $x = 5$ or let $y =$ Green Bay Packers, etc. Often, however, we want to consider many values for the variables. This is typically done

a. state the domains of the variables (e.g., what values can they have)
b. use a quantifier for each variable.

Two important quantifiers

1. the universal quantifier, $\forall$, "for all"

   $P(x)$ = "$3x > 4''$.
   Let **R** be the set of real numbers.
   "$\forall x$ in **R**, $P(x)$" means ...
   Is this true or false?

   $Q(x)$ = "x can vote this year$''$.
   "$\forall x$ who is an American citizen, $Q(x)$" means ...
   Is this true or false?

2. the existential quantifier, $\exists$, "there exists" or "there is some"

   $P(x) = x$ divides 24".
   "$\exists x$, $1 < x < 10$, $P(x)$" means ...
   Is this true or false?

Exercise: Assume that the domain of x is the set of real numbers.

a. Let $P(x) =$ "$2x \geq x$". What is the truth value of $\forall x P(x)$? $\exists x P(x)$?

b. Let $P(x) =$ "$x > x + 1$". What is the truth value of $\forall x P(x)$? $\exists x P(x)$?

Negating propositional functions with quantifiers

Since propositional functions with quantifiers have truth values, we can also negate them.

1. $\neg(\forall x P(x)) \equiv$

   Every Koala can climb.

   $\forall x$ in $\mathbf{R}$, $x^2 > x$.

2. $\neg(\exists x P(x)) \equiv$

   There is a pig that can swim.

   $\exists x$ in $\mathbf{R}$, $x^2 = 2$.

Note: When the domain of the variables is finite and can be listed as $x_1, x_2, \ldots, x_n$, then

(i) $\forall x P(x)$ is the same as

Thus, negating it is

(ii) $\exists x P(x)$ is the same as

Thus, negating it is

Translating English sentences into logical propositions with quantifiers

Let the domain of x and y be all people. Let $P(x)$ be "x is perfect", and $F(y)$ be "y is your friend".

1. Jack is perfect.

2. No one is perfect.

3. Not everyone is perfect.

4. All your friends are perfect.

5. One of your friends is perfect.

6. Everyone is your friend and is perfect.

1.4 Nested Quantifiers

Quantifiers can also be used for predicates with more than one variable. For example, let $C(x, y)$ denote "x has taken class y". Let the domain of x be all UWM students and y be all CS courses. What do these statements mean?

1. $\forall x \forall y C(x, y)$

2. $\forall x \exists y C(x, y)$

3. $\exists x \forall y C(x, y)$

4. $\exists x \exists y C(x, y)$

Key: Read the statement from left to right. The order of the quantifiers matter!

Now, translate these sentences.

1. Nobody has taken all the CS courses at UWM.

2. Each CS class has been taken by at least one student.

3. There is a CS class taken by all students.

Since you understand how to read propositional functions with nested quantifiers, let us determine their truth values.

Let $P(x, y) =$ "$|x| > y$". Let the domains of x and y be integers. For each statement below, determine if they are true or false.

1. $\forall x \forall y P(x, y)$

2. $\forall x \exists y P(x, y)$

3. $\exists y \forall x P(x, y)$

4. $\exists x \forall y P(x, y)$

5. $\forall y \exists x P(x, y)$

6. $\exists x \exists y P(x, y)$

Negating nested quantifiers

What is $\neg \forall x \exists y P(x, y)$?

What is $\neg(\forall y \forall x(P(x, y) \vee Q(x, y)))$?