1 Logic

Definition: A *proposition* is a statement that has a truth value. It is either true or false but not both.

Which of the following statements are propositions?

1. It was very dry this summer.
2. When will this class end?
3. $3^2 + 4^2 = 6^2$.
4. $3x + 5 = 50$.
5. If x is an odd number then so is $3x$.

Note: Mathematicians have adopted the convention that a statement is true provided it is *absolutely true without exception*. Otherwise, it is false. Notice that this may not be the case in casual conversation. Sometimes people will say that a statement is true *and* false. Such ambiguities do not exist in mathematical statements. In the subsequent discussion, we will define more words. Likewise, the definitions are very precise and sometimes different from that of standard usage.

1.1 Logical operators

Compound propositions are built from simpler propositions using logical operators. To understand how these logical operators work, we use *truth tables* to specify the truth value of a compound proposition given the truth values of the simpler propositions.

Let p and q be propositions. We define six very common logical operators.

1. *negation* of p, $\neg p$.
   Let $p =$ "It is cold outside." What is $\neg p$?

2. *conjunction* of p and q, $p \wedge q$, "p *and* q"
   Let $q =$ "I'm wearing boots." What is $p \wedge q$?

The English "or" can have different meanings:

- Prereq: grade C or better in Math 221, 226 or 231; CS 152 or 201.
- You can have tea or coffee with your breakfast.

3. *disjunction* of p and q, $p \vee q$, "*p or q*"
   What is $p \vee q$?

4. *exclusive or* of p and q, $p \oplus q$, "*p xor q*"
   What is $p \oplus q$?

5. *implication* $p \rightarrow q$, "if p then q"

Statement p is called the *hypothesis* while q is called the *conclusion* of the implication.

When is this statement false? *If you mow the lawn, I will pay you $20.*

Transform these statements into an implication:

a. It's hard to get a job without a college degree.

b. To become a professor at Harvard, it is enough to be world-famous.

Common *mathspeak* for $p \to q$ include:

- p implies q
- q whenever p
- p is sufficient for q
- q is necessary for p
- p only if q

The *contrapositive* of $p \to q$ is $\neg q \to \neg p$. The *converse* of $p \to q$ is $q \to p$. What are the contrapositive and converse of these statements?

a. If x is an even number then $3x$ is an even number.

b. If you don't leave by 8 AM, you will be late for class.

6. *biconditional* $p \leftrightarrow q$, "p if and only if q"
   $p \leftrightarrow q$ also means $(p \rightarrow q) \wedge (q \rightarrow p)$.

Common *mathspeak* for $p \leftrightarrow q$ include:

- p is necessary and sufficient for q
- if p then q and conversely

Truth Tables of Compound Propositions

To better understand compound propositions, it is helpful (if it is not too time-consuming) to build their truth tables. One important question to ask is *when do these compound propositions evaluate to true?*

1. $(p \vee \neg q) \rightarrow (p \wedge q)$

2. $(p \rightarrow q) \rightarrow r$

Translating English Sentences to Logical Expressions

To *apply* logic, we need to be able to take English sentences and translate them to logical propositions. Here are the basic steps:

- Parse the compound sentence into propositions without logical connectives.
- Represent each proposition by a letter.
- Combine the propositions with logical connectives used in the sentence.

1. The user has paid the subscription fee but did not enter a valid password.

2. You cannot ride the roller coaster if you are under four feet tall unless you are older than 16 years old.

1.3 Propositional Equivalences

In algebra, you learned that

$$(3x + 2y)(x - y) = 3x^2 - xy - 2y^2.$$

What does it mean for the left-hand side (LHS) of the equation to be equal to the right-hand side (RHS)?

Similar notions exist for propositions.

Definition: Two compound propositions P and Q constructed from the same simple propositions are *logically equivalent* whenever their truth values are the same for *all* the combinations of the truth values of the simple propositions. Notation: $P \equiv Q$ or $P \Leftrightarrow Q$.

Example 1: Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example 2: Show that $(p \rightarrow q) \equiv \neg p \vee q$

Note: Constructing the truth tables for P and Q is a straightforward way of verifying that P and Q are equivalent. The problem, however, is that the table can be very large. *If P is made up of n simple propositions, how many rows does its truth table have?*

Logical Identities

In algebra, the correctness of $(x + 1)^2 = x^2 + 2x + 1$ is obtained not by making sure that the equation holds for *every* value of x but by using various laws such as the distributive law of multiplication over addition and associative law of addition. We have similar laws for propositions.

Here are the identities:

Let us now use these identities to prove the equivalence of propositions.

1. Show that $p \vee (\neg p \wedge q) \equiv p \vee q$.

2. Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$. (That is, an implication is logically equivalent to its contrapositive.)

3. Show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.