4.1 Mathematical Induction

Let $P(n)$ be a statement regarding integer n. In many instances, the goal is to show that $P(n)$ is true for *every* positive integer n. We can of course try to do this using the previous proof methods we've discussed – e.g., direct method, proof by cases, etc. Mathematical induction is one other technique for proving such statements. It is a very powerful tool.

Goal: To prove that $P(n)$ is true for every positive integer n.

Steps:

- *Basis step:* Show $P(1)$ is true.
- *Inductive step:* Show that for every positive integer k, $P(k) \rightarrow P(k+1)$ is true. *That is, for each such k, assume $P(k)$ is true and then show that $P(k+1)$ follows from it.*

Claim: The two steps imply that $P(n)$ is true for all positive integers n.

Example 1: Show that the sum of the first n odd integers is n^2.

What is $P(n)$?

Show that $P(1)$ is true:

State $P(k)$. State $P(k+1)$. Given that $P(k)$ is true, prove that $P(k+1)$ is true.

Example 2: Show that for any positive integer n, $n^3 - n$ is always divisible by 3.

What is $P(n)$?

Show that $P(1)$ is true:

State $P(k)$. State $P(k+1)$. Given that $P(k)$ is true, prove that $P(k+1)$ is true.

Example 3: Show that for every positive integer n, $n < 2^n$.

What is $P(n)$?

Show that $P(1)$ is true:

State $P(k)$. State $P(k+1)$. Given that $P(k)$ is true, prove that $P(k+1)$ is true.

Example 4: Given a sequence of numbers $a_1, a_2, \ldots, a_n$, we say that the sequence has an *inversion* if there are indices i, j such that $i < j$ but $a_i > a_j$. For example, the sequence $3, 8, 2, 5$ has several inversions. What are they?

What kinds of sequences have no inversions? have the maximum number of inversions?

Claim: If $a_1, a_2, \ldots, a_n$ has an inversion, then it has an inversion formed by two consecutive numbers in the sequence. That is, there exists indices i and $i + 1$ so that $a_{i+1} > a_i$.

What is $P(n)$?

For this problem $P(1)$ does not make sense. Thus, we start with the "smallest" integer for which it can be true. Show that $P(2)$ is true.

Assume $P(k)$ is true. Prove that $P(k + 1)$ is true.

Example 5: The Tower of Hanoi. There are three pegs and n disks of varying sizes that can be inserted in a peg. The puzzle starts with discs neatly stacked in one peg from the smallest at the top to the largest at the bottom. The objective is to move the entire stack of disks to another peg using the following rules:

1. Moving a disk means transferring it from one peg to another.
2. Only one disk may be moved at a time.
3. No disk may be placed on top of a smaller disk.

Claim: If there are n disks, $2^n - 1$ moves are sufficient.

Let's prove the claim by induction.

What is $P(n)$?

Prove that $P(1)$ is true.

Assume $P(k)$ is true. Prove that $P(k+1)$ is true.

4.2 Strong Mathematical Induction

Sometimes, we need a *stronger* version of mathematical induction.

Goal: To prove that $P(n)$ is true for every positive integer n.

Steps:

- *Basis step:* Show $P(1)$ is true.
- *Inductive step:* Show that for every positive integer k, this implication is true:

$$[P(1) \land P(2) \land \ldots \land P(k)] \rightarrow P(k+1).$$

  That is, we need to assume that not only is $P(k)$ true, but $P(1), \ldots, P(k-1)$ are also true.

Claim: The two steps imply that $P(n)$ is true for all positive integers n.

Example: (The Fundamental Theorem of Arithmetic.) Every integer greater than 1 can be written as a product of primes.

What is $P(n)$?

Prove that $P(2)$ is true.

Assume $P(2) \land P(3) \land \ldots \land P(k)$ is true. Show that $P(k+1)$ is true.