Connectivity in Graphs

Definition 1 *In a graph G, a path of length k is a sequence of $k+1$ distinct vertices $v_0, v_1, \ldots, v_k$ so that $\{v_i, v_{i+1}\}$ is an edge in G for $i = 0, \ldots, k-1$. A cycle of length k is like a path of length k except its first and last vertices are the same.*

Definition 2 *A graph is connected if there is a path between every pair of vertices in the graph. Furthermore, every graph is the union of its disjoint connected components.*

Definition 3 *A directed graph is strongly connected if for every pair of vertices $\{a, b\}$, there is a path from a to b and b to a. It is weakly connected if the underlying simple graph is connected.*

A characterization of bipartite graphs

Definition 4 *Suppose $G = (V, E)$ is a connected graph. For any two vertices u and v, the distance between u and v, $d(u, v)$, is the length of the shortest path connecting u and v.*

Recall that a graph is *bipartite* if its vertices can be colored red or blue so that no edge has monochromatic endpoints (i.e., both endpoints colored red, or both colored blue).

Lemma 1 *Odd cycles (i.e., cycles with odd length) are not bipartite.*

Theorem 1 *A graph G is bipartite if and only if G does not have an odd cycle as a subgraph.*

Euler Trails and Circuits

MOTIVATION: The Königsberg bridge problem. The townspeople enjoyed walking across the seven bridges in town. They wondered if it's possible to start at some location, travel across each bridge exactly once, and return to the same starting point. In 1736, Leonhard Euler solved this problem by viewing the town layout as a graph.

Let G be a graph. An *Euler trail* in G is a sequence of vertices $v_0, v_1, \ldots, v_k$ so that $\{v_i, v_{i+1}\}$ is and edge for $i = 0, \ldots, k-1$, and each edge of G is visited exactly once. An *Euler circuit* in G is like an Euler trail in G except the first and last vertices are the same.

Which graphs have Euler circuits?

Lemma 2 *Let G be a graph. If G has an Euler circuit, all its vertices have even degrees.*

Proof:

Is the converse true as well?

Lemma 3 *Suppose all the vertices of G have even degrees. If G is connected, then G has an Euler circuit.*

Theorem 2 *A connected graph has an Euler circuit if and only if all its vertices have even degrees.*

What do you think is the corresponding characterization for graphs with Euler trails?