Connectivity in Graphs

Definition 1 In a graph $G$, a path of length $k$ is a sequence of $k + 1$ distinct vertices $v_0, v_1, \ldots, v_k$ so that $\{v_i, v_{i+1}\}$ is an edge in $G$ for $i = 0, \ldots, k - 1$. A cycle of length $k$ is like a path of length $k$ except its first and last vertices are the same.

Definition 2 A graph is connected if there is a path between every pair of vertices in the graph. Furthermore, every graph is the union of its disjoint connected components.

Definition 3 A directed graph is strongly connected if for every pair of vertices $\{a, b\}$, there is a path from $a$ to $b$ and $b$ to $a$. It is weakly connected if the underlying simple graph is connected.
Euler Trails and Circuits

MOTIVATION: The Königsberg bridge problem. The townspeople enjoyed walking across the seven bridges in town. They wondered if it’s possible to start at some location, travel across each bridge exactly once, and return to the same starting point. In 1736, Leonhard Euler solved this problem by viewing the town layout as a graph.

Let $G$ be a graph. An Euler trail in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_k$ so that $\{v_i, v_{i+1}\}$ is an edge for $i = 0, \ldots, k - 1$, and each edge of $G$ is visited exactly once. An Euler circuit in $G$ is like an Euler trail in $G$ except the first and last vertices are the same.

Which graphs have Euler circuits?

**Lemma 2** Let $G$ be a graph. If $G$ has an Euler circuit, all its vertices have even degrees.

Proof:
Is the converse true as well?

**Lemma 3**  *Suppose all the vertices of $G$ have even degrees. If $G$ is connected, then $G$ has an Euler circuit.*
Theorem 2  A connected graph has an Euler circuit if and only if all its vertices have even degrees.

What do you think is the corresponding characterization for graphs with Euler trails?