Graphs

Graphs were first described by the famous mathematician Leonhard Euler to solve the Königsberg bridge problem. Since then, graphs have been used to model problems in many areas including the social sciences, chemistry, engineering, mathematics, and computer science.

Definition 1 *A (simple) graph $G = (V, E)$ is defined by two sets: V, a non-empty set of vertices and E, a set consisting of unordered pairs of vertices from V.*

For example, $V = \{a, b, c, d, e\}$ and $E = \{\{a, c\}, \{b, c\}, \{a, d\}, \{d, e\}, \{c, d\}\}$ describes a graph.

Definition 2 *A directed graph $G = (V, E)$ is defined by two sets: V, a non-empty set of vertices and E, a set consisting of ordered pairs of vertices from V.*

For example, $V = \{a, b, c\}$ and $E = \{(a, b), (b, a), (a, c), (c, b)\}$ describes a directed graph.

Graphs are ubiquitous in Computer Science because they model objects and their relationships with each other; e.g., flowcharts, computer networks, social networks, the webgraph, etc.

More terminologies

• If $e = \{u, v\}$ is an edge in graph G, we say that u and v are *adjacent* or *neighbors*, e is *incident* to u and v, and u and v are *endpoints of* e.

• In a simple graph G, the *degree* of vertex v is $deg(v) =$ number of edges incident to v.

• In a directed graph, the *indegree* of vertex v, $deg^-(v)$, is the number of edges "going into" v while the *outdegree* of vertex v, $deg^+(v)$, is the number of edges "leaving" v.

Lemma 1 *(Handshake Lemma)*
a. In a simple graph $G = (V, E)$*,*

$$\sum_{v \in V} deg(v) = 2|E|.$$

b. In a directed graph $G = (V, E)$*,*

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|.$$

Check:

Proof:

Notes

Suppose the graph G has n vertices.

1. The maximum degree of a vertex in G is ; the minimum degree of a vertex in G is .

2. The maximum number of edges in G is ; the minimum number of edges in G is .

Some special families of graphs

1. *Complete graphs.* In a complete graph, every pair of vertices are adjacent. K_n denotes the complete graph on n vertices.

   degrees of vertices in K_n?

   number of edges in K_n?

2. *Cycles.* In C_n, the cycle has $n \geq 3$ vertices $v_1, v_2, \ldots, v_n$, and v_i is adjacent to v_{i+1} for $i = 1, \ldots, n-1$ and v_n is adjacent to v_1.

   degrees of vertices in C_n?

   number of edges in C_n?

3. *Bipartite Graphs.* A simple graph $G = (V, E)$ is *bipartite* if V can be partitioned into V_1 and V_2 so that every edge in G is between a vertex in V_1 and a vertex in V_2.

   Another way of viewing G as a bipartite graph is its vertices can be colored red or blue so that no two red vertices and no two blue vertices are adjacent.

   Complete bipartite Graphs. $K_{m,n}$ is a bipartite graph with $V = V_1 \cup V_2$ so that $|V_1| = m$ and $|V_2| = n$, and every vertex in V_1 is adjacent to every vertex in V_2.

   degrees of vertices in $K_{m,n}$?

   number of edges in $K_{m,n}$?

New graphs from old

The ideas here are very similar to those of sets.

Definition 3 *A subgraph of graph $G = (V, E)$ is a graph $H = (V', E')$ so that $V' \subseteq V$ and $E' \subseteq E$.*

Definition 4 *A union of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.*

Definition 5 *The complement of graph $G = (V, E)$ is $\bar{G}$ whose vertex set is still V but whose edge set consist of pairs of vertices not present in E. In other words, two vertices are adjacent in $\bar{G}$ if and only if they were not adjacent in G. Hence, $G \cup \bar{G}$ is a complete graph.*

Isomorphism of graphs

Are these graphs alike?

Definition 6 *Two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are isomorphic, denoted as* $G_1 \cong G_2$*, if there is a bijection* $f : V_1 \to V_2$ *so that for any pair of vertices* $a, b, \in V_1$*,* a *and* b *are adjacent in* G_1 *if and only if* $f(a)$ *and* $f(b)$ *are adjacent in* G_2*. (That is, there is a bijection from* V_1 *to* V_2 *that preserves the adjacencies and non-adjacencies of* G_1*.)*

Show that the above example are isomorphic graphs.

When two graphs are isomorphic, they have to have the same number of vertices, the same number of edges, the same kinds of subgraphs, the same distribution of vertex degrees, etc. In other words, they are identical *structurally*. Argue that the following graphs are not isomorphic.