2.3 Functions

**Definition:** Let $A$ and $B$ be sets. A function from $A$ to $B$, $f : A \rightarrow B$, is a rule or an assignment that assigns each element of $A$ to exactly one element of $B$.

*Examples:*

*Non-examples:*

**Definition:** Let $f : A \rightarrow B$ be a function.

* We say that $A$ is the *domain* of $f$ and $B$ is the *co-domain* of $f$.
* Let $a \in A$. If $f(a) = b$ then $a$ is the *pre-image* of $b$ and $b$ is the *image* of $a$.
* The *range* of $f$ is the set $\{b : f(a) = b \text{ for some } a \in A\}$.

*Examples:*

**Definition:** Let $f : A \rightarrow B$ be a function and $S \subseteq A$. The image of $S$ under $f$ is the set $f(S) = \{b : f(s) = b \text{ where } s \in S\}$. 

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Classifying Functions

**Definition:** A function \( f : A \to B \) is **one-to-one** or **injective** if and only if for all \( x, y \in A \), \( f(x) = f(y) \) implies \( x = y \) (or for all \( x, y \in A \), \( x \neq y \) implies \( f(x) \neq f(y) \)).

**Examples:**

**Definition:** A function \( f : A \to B \) is **onto** or **surjective** if and only if for all \( y \in B \), there exists \( x \in A \) so that \( f(x) = y \).

**Examples:**

**Definition:** A function \( f \) is a **bijection** if it is both one-to-one and onto.

**Exercises:** Which functions are one-to-one and/or onto?

I. Picture examples:
II. Descriptions:

1. \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(x) = |x| \).

2. \( f : \{ x : x \text{ is a UWM student} \} \rightarrow \{ y : 0 \leq y \leq 4 \} \) such that \( f(x) \) is the GPA of \( x \).

3. \( f : \{ x : x \text{ is a VISA card holder} \} \rightarrow \{ y : y \text{ is a 16 digit number} \} \) such that \( f(x) \) is the VISA card number of \( x \).

4. \( f : \{ x : x \in \mathbb{Z}, 0 \leq x \leq 63 \} \rightarrow \{ y : y \text{ is a 0-1 string of length 6} \} \) such that \( f(x) \) is the binary representation of \( x \).
Showing a function is one-to-one, onto, or bijective

Let $f : A \to B$ be a function.

- **To show $f$ is one-to-one:** Argue that for any $x, y \in A$ the implication $f(x) = f(y)$ implies $x = y$ or its contrapositive $x \neq y$ implies $f(x) \neq f(y)$ is true.

- **To show $f$ is onto:** Argue that for each $y \in B$, there is an $x \in A$ so that $f(x) = y$. That is, identify the pre-image of $y$.

- **To show $f$ is a bijection:** Prove $f$ is one-to-one and onto.

*Example:* Distributing $n$ identical balls to $m$ distinct boxes.
Theorem 1 Suppose $f : A \to B$ is a function. We can compare the cardinalities of $A$ and $B$ if we know that $f$ is one-to-one, onto, or bijective.

(1) If $f$ is one-to-one, then

(2) If $f$ is onto, then

(3) If $f$ is a bijection then

Inversions and Compositions

Definition: Let $f : A \to B$ be a bijection. The \textit{inverse} of $f$, $f^{-1}$, is a function from $B$ to $A$ so that $f^{-1}(b) = a$ if and only if $f(a) = b$.

Example:

\textit{Note:} A bijection is \textit{invertible} because its inverse always exists.
**Definition:** Let $f_1: A \to B$ and $f_2: B \to C$ be functions. The *composition of $f_1$ and $f_2$* is the function $f_2 \circ f_1: A \to C$ where $f_2 \circ f_1(a) = f_2(f_1(a))$.

*Example:*