

2.3 Functions

Definition: Let A and B be sets. A *function from A to B* , $f : A \rightarrow B$, is a rule or an assignment that assigns each element of A to exactly one element of B .

Examples:

Non-examples:

Definition: Let $f : A \rightarrow B$ be a function.

- We say that A is the *domain* of f and B is the *co-domain* of f .
- Let $a \in A$. If $f(a) = b$ then a is the *pre-image* of b and b is the *image* of a .
- The *range* of f is the set $\{b : f(a) = b \text{ for some } a \in A\}$.

Examples:

Definition: Let $f : A \rightarrow B$ be a function and $S \subseteq A$. The image of S under f is the set $f(S) = \{b : f(s) = b \text{ where } s \in S\}$.

Classifying Functions

Definition: A function $f : A \rightarrow B$ is *one-to-one* or *injective* if and only if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$ (or for all $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$).

Examples:

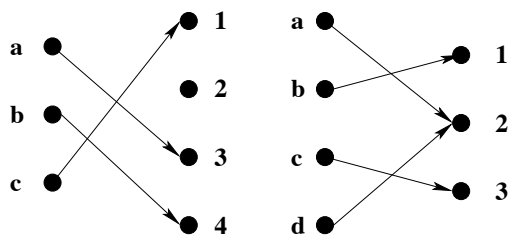
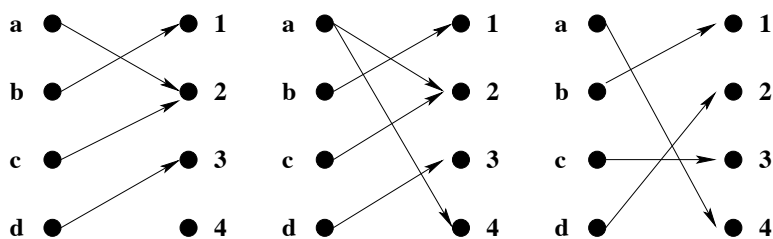
Definition: A function $f : A \rightarrow B$ is *onto* or *surjective* if and only if for all $y \in B$, there exists $x \in A$ so that $f(x) = y$.

Examples:

Definition: A function f is a *bijection* if it is both one-to-one and onto.

Exercises: Which functions are one-to-one and/or onto?

I. Picture examples:



II. Descriptions:

1. $f : \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x) = |x|$.
2. $f : \{x : x \text{ is a UWM student}\} \rightarrow \{y : 0 \leq y \leq 4\}$ such that $f(x)$ is the GPA of x .
3. $f : \{x : x \text{ is a VISA card holder}\} \rightarrow \{y : y \text{ is a 16 digit number}\}$ such that $f(x)$ is the VISA card number of x .
4. $f : \{x : x \in \mathbf{Z}, 0 \leq x \leq 63\} \rightarrow \{y : y \text{ is a 0-1 string of length 6}\}$ such that $f(x)$ is the binary representation of x .

Showing a function is one-to-one, onto, or bijective

Let $f : A \rightarrow B$ be a function.

- *To show f is one-to-one:* Argue that for any $x, y \in A$ the implication $f(x) = f(y)$ implies $x = y$ or its contrapositive $x \neq y$ implies $f(x) \neq f(y)$ is true.
- *To show f is onto:* Argue that for each $y \in B$, there is an $x \in A$ so that $f(x) = y$. That is, identify the pre-image of y .
- *To show f is a bijection:* Prove f is one-to-one *and* onto.

Example: Distributing n identical balls to m distinct boxes.

Theorem 1 Suppose $f : A \rightarrow B$ is a function. We can compare the cardinalities of A and B if we know that f is one-to-one, onto, or bijective.

(1) If f is one-to-one, then

(2) If f is onto, then

(3) If f is a bijection then

Inversions and Compositions

Definition: Let $f : A \rightarrow B$ be a bijection. The *inverse of f* , f^{-1} , is a function from B to A so that $f^{-1}(b) = a$ if and only if $f(a) = b$.

Example:

Note: A bijection is *invertible* because its inverse always exists.

Definition: Let $f_1 : A \rightarrow B$ and $f_2 : B \rightarrow C$ be functions. The *composition of f_1 and f_2* is the function $f_2 \circ f_1 : A \rightarrow C$ where $f_2 \circ f_1(a) = f_2(f_1(a))$.

Example: