6.1 The Basics of Counting

Here's our basic goal: given a set S, determine $|S|$, the size of S. Many counting problems can be solved by applying one or a combination of the principles below. If you read them carefully, notice that most of them are not new; we've considered them in our discussions on sets and functions.

- **The brute force method**. Enumerate all the elements of S and count them!

  When $|S|$ is large, however, this is usually not a feasible approach. Not only does enumeration take a long time, it is also tricky to determine if one has in fact listed all the elements of S.

- **The sum method**. Suppose $S = A_1 \cup A_2 \cup \ldots \cup A_m$ and $A_i \cap A_j = \emptyset$ for every pair of i, j. Then
$$|S| = |A_1| + |A_2| + \ldots + |A_m|.$$
  If $S = A_1 \cup A_2$, then $|S| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- **The product method**. Suppose $S = A_1 \times A_2 \times \ldots \times A_m$. Then
$$|S| = |A_1| \times |A_2| \times \ldots \times |A_m|.$$

- **The complementation method.** Suppose that $S \subseteq U$ where U is some universal set. Then
$$|S| = |U| - |\bar{S}|.$$

- **The bijection method.** Suppose that there exists a bijection $f : S \to T$. Then
$$|S| = |T|.$$

Some exercises:

1. When a red and green dice are rolled, how many ways can we get a sum of 7?

2. Suppose there are five Spanish books, six French and seven German books. If a book must be chosen among these, how many choices are there?

3. Suppose there are five Spanish books, six French and seven German books. How many ways are there to pick two books, not both in the same language?

4. How many bit strings (0/1-strings) are there of length n?

5. How many subsets does a set of size n have?

6. A *palindrome* is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

7. A state's license plate consists of six characters. The first three must be letters while the last three are numbers.

   a. How many license plates can a state issue?

   b. How many does not contain the letter z?

   c. How many contains at least one vowel?

The Product Rule. Suppose that a procedure can be broken down into a sequence of m stages so that after stages $1, 2, \ldots, i-1$ are completed there are n_i ways of doing stage i for $i = 1, \ldots, m$. Then the procedure has $n_1 \times n_2 \times \ldots \times n_m$ different results.

Example 1. Suppose a photographer wishes to take picture of six people, two of whom are Alice and Bob.

- How many ways are there of arranging the six people in a line?

- so that Alice is at one end of the line and Bob is at the other end?

- so that Alice and Bob are next to each other?

Example 2. How many ways are there to form a 3-letter sequence using a, b, c, d, e where the letters can be repeated and e occurs at least once in the sequence?

Approach 1: To form the 3-letter sequence, first choose a position for e. Then fill the remaining positions with any of the five letters. Using the product rule, what is the number of sequences?

Approach 2: Use the complementation method. First, count the number of 3-letter sequences that can be formed with five letters. Of these, count the number of 3-letter sequences that do not use e. What is the number of 3-letter sequences that use e?

Which one is correct?

The Revised Product Rule. Suppose that a procedure can be broken down into a sequence of m stages so that after stages $1, 2, \ldots, i-1$ are completed there are n_i ways of doing stage i for $i = 1, \ldots, m$. Furthermore, *suppose different choices always lead to different results.* That is, whenever $(w_1, w_2, \ldots, w_m)$ and $(w'_1, w'_2, \ldots, w'_m)$ describe choices made at each of the m stages and $(w_1, w_2, \ldots, w_m) \neq (w'_1, w'_2, \ldots, w'_m)$, different outcomes are produced. Then the procedure has $n_1 \times n_2 \times \ldots \times n_m$ different results.

Key: When you are breaking a procedure into stages, make sure that different choices always lead to different results. Otherwise, overcounting can occur.

6.3 Permutations and Combinations

Definition: For a positive integer n, $n! = 1 \times 2 \times \ldots \times n$. By convention $0! = 1$.

Definition: Let S be a set. A *permutation* of S is an ordered arrangement of all the elements of S. An *r-permutation* of S is an ordered arrangement of r elements of S.

Example. Suppose $S = \{1, 2, 3, 4, 5\}$. Write down some permutations of S and some 3-permutations of S.

Definition: Let $P(n, r)$ denote the number of r-permutations of an n-element set.

Questions: What is $P(n, n)$? $P(n, 1)$? $P(n, 0)$?

Theorem 1 $P(n, r) =$

Proof:

Example. A group consisting of n men and n women went out to see a movie. How many ways are there of

- arranging them all in a row?
- so that the men and women alternate?
- so that all the women are seated together?

Definition: An r-*combination* of S is a size-r subset of S. That is, it is an *unordered* selection of r elements of S.

Example. Suppose $S = \{1, 2, 3, 4, 5\}$. Write down all the 5-combinations and 4-combinations of S.

Definition: Let $C(n, r)$ denote the number of r-combinations of an n-element set.

Questions: What is $C(n, n)$? $C(n, 1)$? $C(n, 0)$?

Theorem 2 $C(n, r) =$

Proof:

Corollary 1 *Let n and r be non-negative integers with $r \leq n$. Then*

$$C(n, r) = C(n, n - r).$$

Proof:

Example. There are seven women and four men eligible for committee membership. One of the women is Alice; one of the men is Bob. How many ways are there to form the committee if

- the committee has to have five people?
- the committee has to have three women and two men?
- the committee has to have five people, at least two of which are women?
- the committee has to have five people, and Alice or Bob should be part of the committee?

Example. How many 5-card poker hands obtained from a 52-deck card have the following:

- four aces?
- four of a kind?
- a full house (3-of-a-kind and a pair)?
- a straight (a set of five consecutive values)?

6.5 Generalized Permutations and Combinations

There are different kinds of counting problems considered in this section. We will examine only two: distributing identical balls into distinct boxes, and distributing distinct balls into distinct boxes.

Distributing indistinguishable balls into distinct boxes

Given m identical balls and n distinct boxes, we want to distribute all the balls into the boxes so that what matters is how many balls each box contains. How many possible distributions are there?

1. A bagel shop sells eight varieties of bagels. How many ways are there to choose

   a. a dozen bagels?

   b. a dozen bagels with at least one of each kind?

   c. a dozen bagels with at least three egg bagels?

   d. a dozen bagels with at least three egg bagels and no more than two salty bagels?

2. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

   where each x_i is an integer and

a. $x_i \geq 0$ for $i = 1, \ldots 5$?

b. and $0 \leq x_1 \leq 10$?

c. and $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 4$?

Distributing distinct balls into distinct boxes

Given m *distinct* balls and n distinct boxes, again, we want to distribute (not necessarily all) the balls into the boxes so that there are k_i in box i for $i = 1, \ldots n$. How many distributions are there?

1. How many ways are there to distribute hands of five cards to each of four players from the standard deck of 52 cards?

2. How many different strings can be made by re-ordering the letters of the word SUCCESS?

Circular Arrangements

Example. When arranging guests around a circular table, it is reasonable to say that two arrangements are "the same" if one is simply a rotation of the other. If there are five guests, how many distinct ways can they be arranged around a circular table?

Theorem 3 *Suppose two arrangements in a circle are considered identical if one can be obtained from the other by rotation. The number of distinct arrangements of n objects in a circle is*

Example. Recall the six people from the picture-taking session that included Alice and Bob.

- How many ways can they be seated around a circular table?
- so that Alice and Bob are next to each other?
- so that Alice and Bob are not next to each other?
- What if there are three men and three women in the group and all the women have to be seated next to each other?