Random Variables, Expectations

In a previous example, we considered the situation where two dice are rolled and asked what is the probability that the sum of the numbers is 4. We answered this by letting E denote the event that the sum of the numbers is 4 so that $E = \{(1,3),(2,2),(3,1)\}$. Then we noted that $p(E) = |E|/|S| = 3/36 = 1/12$.

An alternative is to define a function $X : S \to \mathbf{R}$ so that $X(s)$ indicates the sum of the numbers in the outcome s, and then ask what is the probability that $X = 4$ (i.e., what is $P(X = 4)$?). The function X is an example of a *random variable.*

Definition A *random variable* is a function $X : S \to \mathbf{R}$. That is, a random variable assigns a real number to each possible outcome.

Describing events. Events can be described by specifying what the X-values of the outcomes should be. For example, we use $(X = r)$ to describe the set containing all outcomes s such that $X(s) = r$; i.e., $(X = r) = \{s : X(s) = r\}$. Similarly, $(X \leq r) = \{s : X(s) \leq r\}$, etc.

Example 1. Suppose a coin is tossed five times. For each outcome s, let $X_H(s)$ denote the number of heads in s; e.g., $X_H(HTTHT) = 2$, $X_H(TTTTT) = 0$, etc. List the outcomes in the event ($X_H \leq 1$ or $X_H \geq 4$).

What is $P(X_H \leq 1$ or $X_H \geq 4)$?

Example 2. Suppose a coin is tossed until a head appears. For each outcome s, let $X(s)$ denote the number of tosses in s. For example, $X(TTTH) = 4$. Describe the outcomes in the event $(X \geq 4)$.

What is $P(X \geq 4)$?

Definition The *distribution of random variable X over sample space S* describes $P(X = r)$ for all $r \in \mathbf{R}$ such that $P(X = r) \neq 0$.

Example 1 cont'd. What is the distribution of X_H?

Binomial distributions $B(n, p)$

Example 2 cont'd. What is the distribution of X?

Geometric distributions $G(p)$

Expectation

Definition The *expected value or mean* of a random variable X is defined as

$$E[X] = \sum_{s \in S} X(s)p(s).$$

- $E[X]$ is a one-number summary of X. It gives us an indication of what the value of X will be most of the time – how small or large it is.
- $E[X]$ is also called the "average" value of X. It stems from the following property. Suppose the experiment associated with X is repeated n times, each one indpendent of the others. Let the observed values of X be $x_1, x_2, \ldots, x_n$. The Law of Large Numbers states that if $E[X] < \infty$, then

$$\frac{\sum_{i=1}^{n} x_i}{n} \to E[X].$$

Example. Suppose we toss a fair coin three times. Let X_H denote the number of heads that appear. What is $E[X_H]$?

Using the definition of $E[X]$ to compute its value can be tedious because we have to consider *all* outcomes $s \in S$. Another option is to group outcomes according to their X-values and proceed accordingly.

Lemma Let X be a random variable. Then

$$E[X] = \sum_{r \in \mathbf{R}} rP(X = r).$$

Example cont'd. For the experiment where we tossed a fair coin three times. Use the new formula to compute $E[X_H]$.

Example 2 cont'd. Suppose we toss a fair coin until a head appears. Let $X(s)$ denote the number of tosses in s. What is $E[X]$?

Example. Recall that in the Pick-6 lottery, six numbers are chosen from $\{1, 2, \ldots, 40\}$ without replacement. It pays \$1,000,000 if you have the winning ticket and \$0 otherwise. Suppose you bought one ticket. What is your expected payoff? That is, let X denote the amount of money received by a single ticket holder – who presumably chose his or her numbers at random. What is $E[X]$?

Theorem 1 Linearity of expectations.
(i) Let X_i, $i = 1, \ldots, n$ be random variables on sample space S. Then

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n].$$

(ii) Let X be a random variable on S and $a, b \in \mathbf{R}$. Then

$$E[aX + b] = aE[X] + b.$$

Example cont'd. Returning to our experiment where a fair coin is tossed three times, here's a third way of computing X_H. Let X_i equal 1 if the coin turned up heads during the ith toss and 0 otherwise. Thus, $X_H = X_1 + X_2 + X_3$.

What is $E[X_1]$? $E[X_2]$? $E[X_3]$? Based on your answers and the linearity of expectation, compute $E[X]$.

The Hatcheck problem. Suppose n men checks in their hats in a restaurant. Unfortunately, their claim stubs get randomly mixed up so that a man is equally likely to receive another man's stub. When they go claim their hats, on average how many will get their own hats back?

Inversions in a random sequence. Recall that in a sequence of n numbers: $a_1, a_2, \ldots, a_n$, a_i and a_j form an *inversion* if $i < j$, $a_i > a_j$. Suppose that the sequence was randomly arranged. What is the expected number of inversions in the sequence?

Variance

Given a random variable X, its expectation $E[X]$ tells us its average value but it does *not* capture how widely the values of X vary. For example, let $X_1 = 1000$ with probability 1 and let $X_2 = 0$ with probability 1/2 and 2000 with probability 1/2. Both X_1 and X_2 have expectations equal to 1000 but clearly X_2 is more "spread out" that X_1. To quantify the spread of a random variable, we define the notion of *variance*.

Definition Let X be a random variable on sample space S. The *variance of* X is

$$var(X) = \sum_{s \in S} (X(s) - E[X])^2 p(s).$$

The *standard deviation of* X is $\sqrt{var(X)}$.

Example. What is $var(X)$ if X is the number that comes up when a die is rolled?

Here's a less tedious way of computing $var(X)$.

Theorem 2 *Let X be a random variable on sample space S. Then $var(X) = E[X^2] - E[X]^2$.*

Example. What is $var(X_1)$? $var(X_2)$? (X_1 and X_2 are the random variables described in the introduction.

Example. Recompute $var(X)$, X is the number that comes up when a die is rolled, using the new formula.