Basic Bayes’ Theorem

Suppose $E$ and $F$ are events from a sample space $S$ such that $p(E) > 0$ and $p(F) > 0$. Since $E \cap F$ and $E \cap F'$ are disjoint and $E = (E \cap F) \cup (E \cap F')$, we have

$$p(E) = p(E \cap F) + p(E \cap F').$$

If conditional probabilities are known, we can use them to determine $p(E \cap F)$ and $p(E \cap F')$. That is, since

$$p(E \cap F) = p(E|F)p(F)$$
$$p(E \cap F') = p(E|F')p(F'),$$

we have

$$p(E) = p(E|F)p(F) + p(E|F')p(F').$$

Example: Suppose that in some alternate universe your life is in danger and you are facing a judge. The judge hands you 50 red balls, 50 blue balls and two opaque jars (i.e., you can’t see the inside of the jars). Your job is to distribute the 100 balls into the two jars. But before you do so, he lets you know what will happen afterwards. He’ll pick one of the two jars uniformly at random. He’ll then pick a ball from the chosen jar. If the ball is red, you’ll live; if it is blue, you’ll die. So how should you distribute the 100 balls into the two jars so you maximize your chances of staying alive?
Bayes’ Theorem makes use of the ideas above to compute $p(F|E)$.

**Theorem 0.1.** Suppose $E$ and $F$ are events from a sample space $S$ such that $p(E) > 0$ and $p(F) > 0$. Then

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}.$$

**Proof:**

**Example:** Suppose that one person in 100,000 has a particular rare disease. Researchers have developed an diagnostic test for this disease with the following statistics: the test is correct 99.0% of the time when given to a person who has a disease; it is correct 99.5% of the time when given to a person who does not have the disease. Given this information, compute

a. the probability that a person who tests positive for the disease has the disease, and
b. the probability that a person who tests negative for the disease does not have the disease.

Should a person who tests positive be very concerned that he or she has the disease?