Instructions: There are five problems in this exam. Make sure you look through them carefully. Please write legibly and justify all your answers.

1. (3 pts.) Recall that a tautology is a statement that is always true. Using propositional identities, show that \( \neg(p \to q) \to p \) is a tautology.
2. (5 pts.) Let $P(x, y)$ be the statement “Student $x$ in Class $y$ knows calculus.” Express each of these statements using quantifiers and $P(x, y)$:

a. Some students in English 101 know calculus.

b. Not every student in Physics 101 knows calculus.

c. The only student that is in both English 101 and Physics 101 that knows calculus is Johnny.

d. Every class has a student in it who knows calculus.

e. There is at least one class with no students who know calculus.
3. (5 pts.) Consider the following premises:

P1: \((p \land q) \rightarrow r\)

P2: \(p \lor s\)

P3: \(\neg q \rightarrow t\)

P4: \(u \rightarrow \neg(s \lor t)\)

P5: \(\neg r\)

Using the rules of inference, show that the five premises imply \(\neg u\).

(Note: For one of the implications above, you may want to convert it to another proposition using the implication rule.)
4. (3 pts.) About Proving.
   
a. Prove that the sum of three consecutive numbers is always divisible by 3.
   
b. On the other hand, prove that the sum of four consecutive numbers is *never* divisible by 4.
   
   (Hint: If $a$ is the first number, how do you express the numbers after it?)
5. (5 pts.) Let $U$ be a set with $n$ elements: $x_1, x_2, \ldots, x_n$. Let $S$ be a subset of $U$. It is often convenient to use a bit string of length $n$ to represent $S$ by using the following function:

$$f : \mathcal{P}(U) \rightarrow \text{set of bit strings of length } n,$$

such that $f(S) = b_1b_2\cdots b_n$ where $b_i = 1$ if $x_i \in S$ and $b_i = 0$ if $x_i \notin S$.

For example, suppose $U = \{1, 2, 3, 4, 5\}$. The subset $\{1\}$ is represented by the bit string 10000 while the subset $\{2, 5\}$ is represented by the bit string 01001. For a general set $U = \{x_1, x_2, \ldots, x_n\}$, please answer the following questions:

a. What is $f(U)$? $f(\emptyset)$?

b. What is the relationship between $f(S)$ and $f(\bar{S})$? That is, how are the bit strings representing $S$ and its complement $\bar{S}$ related?

c. Now that you have a “feel” for this function, show that $f$ is one-to-one.

d. Show $f$ is onto.

e. From c and d, we know that $f$ is a bijection. Describe $f^{-1}$. Make sure that you state its domain and co-domain.