A simple rule for checking when a number is divisible by 3

Let $x$ be an integer. What is a quick way to determine if $x$ is divisible by 2 or 5?

Here’s a simple rule for determining if $x$ is divisible by 3: an integer $x$ is divisible by 3 if and only if the sum of its digits is also divisible by 3. Let’s try some examples.

Our goal is to prove that this rule is correct. Here’s a proof-friendlier way of stating the rule:

**Theorem 1** Let $x = a_k \times 10^k + a_{k-1} \times 10^{k-1} + \ldots + a_1 \times 10^1 + a_0$ where each $a_i$ is an integer between 0 and 9 and $a_k \neq 0$. Then $x$ is divisible by 3 if and only if $a_k + a_{k-1} + \ldots + a_1 + a_0$ is also divisible by 3.

First, we establish a basic fact:

**Lemma 1** For every positive integer $n$, $10^n - 1$ is divisible by 3.

Part 1: if $a_k + a_{k-1} + \ldots + a_0$ is divisible by 3 then so is $x$.

Part 2: if $x$ is divisible by 3 then so is $a_k + a_{k-1} + \ldots + a_0$. 

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