Homework 9 Solutions

1. (i) a. The set of all 5-permutations of 30 objects.

   $S = P(30,5) = 17,100,720$

   b. The set of all 5-combinations of 30 objects.

   $|S| = \binom{30}{5} = 142,506$

   c. The set of all 5-permutations of 30 objects with repetition.

   $|S| = 30^5 = 24,300,000$

   (ii) Note that under all three interpretations, each outcome is equally likely. This allows us to simply consider the size of an event over the size of the sample space. Let E = the event that you don't draw a marble numbered 5.

   a.

   $$|E| = P(29,5) = 14,250,600$$
   $$Pr(E) = \frac{|E|}{|S|} = \frac{14,250,600}{17,100,720} = 0.8\overline{333}$$

   b.

   $$|E| = \binom{29}{5} = 118,755$$
   $$Pr(E) = \frac{|E|}{|S|} = \frac{118,755}{142,506} = 0.8\overline{333}$$

   c.

   $$|E| = 29^5 = 20,511,149$$
   $$Pr(E) = \frac{|E|}{|S|} = \frac{20,511,149}{24,300,000} \approx 0.8441$$

   (iii) Let E = the event that at least one of the marbles drawn is numbered 16, 17, 18, 19, or 20. Keep in mind that $Pr(E) = 1 - Pr(\overline{E})$.

   a.

   $$|\overline{E}| = P(25,5) = 6,375,600$$
   $$Pr(E) = \left(1 - Pr(\overline{E})\right) = \left(1 - \frac{|\overline{E}|}{|S|}\right) = \left(1 - \frac{6,375,700}{17,100,720}\right) \approx 0.6272$$

b.

$$|\overline{E}| = C(25,5) = 53,130$$
$$Pr(E) = \Big(1 - Pr(\overline{E})\Big) = \Big(1 - \frac{|\overline{E}|}{|S|}\Big) = \Big(1 - \frac{53,130}{142,506}\Big) \approx 0.6272$$

c.

$$|\overline{E}| = 25^5 = 9,765,625$$
$$Pr(E) = \Big(1 - Pr(\overline{E})\Big) = \Big(1 - \frac{|\overline{E}|}{|S|}\Big) = \Big(1 - \frac{9,765,625}{24,300,000}\Big) \approx 0.5981$$

2. Let E_i be the event that the number i is rolled. Keep in mind that $\sum_{i=1}^{5} E_i$ must equal 1 in order to have a valid probability distribution.

$$Pr(E_1) = Pr(E_5) = \frac{6}{19}$$
$$Pr(E_2) = Pr(E_4) = \frac{3}{19}$$
$$Pr(E_3) = \frac{1}{19}$$

Let $F =$ the event that a perfect square is rolled. In other words, $F = E_1 \cup E_4$. Since E_1 and E_4 are disjoint events (i.e. they cannot occur at the same time) we simply sum the probability that each event occurs individually. Were E_1 and E_4 not disjoint, we would need to apply the inclusion-exclusion principle.

$$Pr(F) = Pr(E_1) + Pr(E_4) = \frac{6}{19} + \frac{3}{19} = \frac{9}{19}$$

3. We are going to think of outcomes as *ordered* die rolls. This might be counter to your intuition since the dice are thrown simultaneously, but consider the following: if we assigned each die a unique color, we could assign each outcome to a corresponding ordered outcome. For example, we could say that the "blue" die corresponds to "die #1" and the "green" die corresponds to "die #2" and so on.

What's nice about this ordered interpretation is that each outcome can then be thought of as a string of integers, where the i^{th} integer in the string is the number that appeared on the i^{th} die. Since we're rolling five dice, each string will be of length five and each integer will be in the range $[1,6]$. We also know that the number appearing on each die is independent of what numbers appear on the other dice, so each possible string is equally likely. Now when we're asked "what's the probability that we roll a four of a kind?" we know that we're actually being asked "how many strings, out of all possible strings, contain four identical integers?"

In each case, let E = the event described, e.g. all 1's, four of a kind, etc. Note that $Pr(E) = \frac{|E|}{|S|}$ and $|S| = 6^5 = 7776$. Also, be aware that "four of a kind" means *exactly* four occurrences of a single number, and similarly for the other events.

(i) E contains only one string, the string of all 1's.

$|E| = 1$

$$Pr(E) = \frac{1}{7776}$$

(ii) Let E_i be the event that four i's are rolled.

$$Pr(E) = Pr(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6)$$

Notice that all E_i are pairwise disjoint (no two can occur at the same time). By union of disjoint events:

$$Pr(E) = Pr(E_1) + Pr(E_2) + Pr(E_3) + Pr(E_4) + Pr(E_5) + Pr(E_6)$$

Also notice that $Pr(E_i) = Pr(E_j)$ for all $i, j \in [1, 6]$, i.e. it's just as likely to get four of one number as it is to get four of any other. So we can say:

$$Pr(E) = 6 \times Pr(E_1)$$

Consider the following 2 step procedure for generating an appropriate integer string:

(1) Choose the number that is not a 1. $\binom{5}{1}$

(2) Choose the position of this number. $\binom{5}{1}$

$$|E_1| = \binom{5}{1}\binom{5}{1} = 25$$

$$Pr(E_1) = \frac{25}{7776}$$

There's another way we could have computed $Pr(E_1)$, as well. If we call rolling a 1 a "success" and rolling anything else a "failure," then $Pr(E_1)$ becomes the probability of getting exactly four successes in five Bernoulli trials where $p = \frac{1}{6}$. We have a formula for this:

$$Pr(E_1) = \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1$$

In either case, $Pr(E_1) \approx 0.0032$. Now we can finally compute $Pr(E)$.

$$Pr(E) = 6 \times Pr(E_1) \approx 0.0193$$

(iii) Consider the following 4 step procedure for generating an appropriate integer string:

(1) Choose two numbers for the pairs. $\binom{6}{2}$

(2) Choose the positions for the first pair. $\binom{5}{2}$

(3) Choose the positions for the second pair. $\binom{3}{2}$

(4) Choose the last number. $\binom{4}{1}$

$$|E| = \binom{6}{2}\binom{5}{2}\binom{3}{2}\binom{4}{1} = 1800$$

$$Pr(E) = \frac{1800}{7776} \approx 0.2315$$

(iv) Choose the positions of the five numbers (i.e., permute them).

$$|E| = 5! = 120$$

$$Pr(E) = \frac{120}{7776} \approx 0.0154$$

(v) Choose the five numbers and their positions.

$$|E| = \binom{6}{5}5! = 720$$

$$Pr(E) = \frac{720}{7776} \approx 0.0926$$

7.2.26. $|S| = 2^3 = 8$

$$Pr(E) = Pr(\text{exactly three 1's}) + Pr(\text{exactly one 1}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$Pr(F) = \frac{1}{2}$$

$$Pr(E \cap F) = \frac{1}{4}$$

$$Pr(E)Pr(F) = \frac{1}{4}$$

Since $Pr(E \cap F) = Pr(E)Pr(F)$, E and F and independent.

7.2.30. a. $(0.5)^{10} = \frac{1}{1024}$

b. $(0.6)^{10} \approx 0.0060466$

c. $\prod_{i=1}^{10} \frac{1}{2^i} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\cdots\left(\frac{1}{1024}\right) = \frac{1}{2^{55}}$

7.3.2. First we solve for $Pr(E \cap F)$.

$$Pr(F \mid E) = \frac{Pr(E \cap F)}{Pr(E)}$$
$$Pr(E \cap F) = Pr(F \mid E)Pr(E) = \left(\frac{5}{8}\right)\left(\frac{2}{3}\right) = \frac{5}{12}$$

Now we solve for $Pr(E \mid F)$.

$$Pr(E \mid F) = \frac{Pr(E \cap F)}{Pr(F)} = \left(\frac{5}{12}\right)\left(\frac{4}{3}\right) = \frac{5}{9}$$

7.3.4. Let $E =$ the event that Ann picked a ball from the second box and let $F =$ the event that Ann picked an orange ball.

$$Pr(F) = \left(\frac{1}{2}\right)\left(\frac{3}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{11}\right) = \frac{34}{77}$$
$$Pr(E \cap F) = \left(\frac{1}{2}\right)\left(\frac{5}{11}\right) = \frac{5}{22}$$
$$Pr(E \mid F) = \frac{Pr(E \cap F)}{Pr(F)} \approx 0.5147$$