

Homework 3

Due 09.27.11 (Thursday)

1. Sec. 1.8: 4
2. Sec. 6.2: 6, 14
3. It is easy to determine if an integer x is divisible by 2: Check x 's last digit. If it is even, x is divisible by 2; otherwise, it isn't. A similar rule exists for determining if x is divisible by 5. If x 's last digit is 0 or 5, x is divisible by 5; otherwise it isn't. In this problem, we want to prove a rule for determining when x is divisible by 3 that is of a slightly different variety. We wish to show that *an integer x is divisible by 3 if and only if the sum of its digits is also divisible by 3*. For example, 863 is not divisible by 3 because $8 + 6 + 3 = 17$ and 17 is not divisible by 3. On the other hand, 873 is because $8 + 7 + 3 = 18$ and 18 is divisible by 3. Notice that this is a very handy rule when x is large since adding the digits of x and dividing the sum by 3 is easier than actually dividing x by 3.

To show that the rule is correct, we need to establish a few facts first. It is your job to prove them.

- a. For every positive integer n , $10^n - 1$ is divisible by 3. (Hint: Try substituting different values for n . What does $10^n - 1$ look like?)
- b. Let s_0, s_1, \dots, s_k be integers and $s = s_0 + s_1 + \dots + s_k$. If s is divisible by 3 and s_0, s_1, \dots, s_{k-1} are divisible by 3, then s_k is also divisible by 3.

Let us now tackle the rule. Since we are considering the digits of x , it is important that we make these digits explicit. Hence, let us write x as

$$x = a_k \times 10^k + a_{k-1} \times 10^{k-1} + \dots + a_1 \times 10^1 + a_0$$

where each a_i is an integer between 0 and 9 and $a_k \neq 0$. For example, if $x = 863$ then $x = 8 \times 10^2 + 6 \times 10^1 + 3$ so that $a_2 = 8, a_1 = 6, a_0 = 3$. Now, rewrite x as follows:

$$x = a_k \times (10^k - 1) + a_{k-1} \times (10^{k-1} - 1) + \dots + a_1 \times (10^1 - 1) + a_k + a_{k-1} + \dots + a_0.$$

- c. Using parts a and b above, prove that if x is divisible by 3 then so is $a_k + a_{k-1} + \dots + a_0$, the sum of its digits.
- d. This time around, prove the converse. That is, show that if $a_k + a_{k-1} + \dots + a_0$ is divisible by 3 then so is x .

Here's hoping you'll remember the rule and put it to good use!

4. In Computer Science, *Satisfiability* is the name that is used to describe a broad class of problems that goes like this: Let C be a compound proposition that is constructed from the variables x_1, x_2, \dots, x_n . Determine if there is a truth assignment for x_1, x_2, \dots, x_n so that C evaluates to true. If such a truth assignment exists, then C is said to be *satisfiable* and the truth assignment *satisfies* C . Many problems can be framed as a satisfiability problem. For example, Section 1.3 of your book shows how the puzzle Sudoku can be cast as a satisfiability problem on 729 variables!

For this problem, we are interested in a specific class of Satisfiability problems called 2-SAT, which is short for 2-satisfiability. Below is an instance of 2-SAT:

$$C = (x_1 \vee \bar{x}_2) \wedge (x_2 \vee \bar{x}_3) \wedge (x_3 \vee \bar{x}_4).$$

A *literal* is a variable (e.g., x_i) or a negation of a variable (e.g., \bar{x}_i). In a 2-SAT instance, each *clause* – those enclosed in parenthesis – is made up of two literals connected by an “or” operator. The clauses are then connected by “and” operators. Our goal is to show that when a 2-SAT instance is satisfiable, the truth assignments that satisfy the instance obey a nice property.

In the above example, the truth assignment $x_1 = T, x_2 = T, x_3 = T$ and $x_4 = F$ satisfies C .

- a. Find two other truth assignments that satisfy C .
- b. Let us call these three truth assignments A_1, A_2 and A_3 . Form a fourth truth assignment A_4 as follows: For each x_i , set its value to its majority value in A_1, A_2, A_3 . For example, if x_1 's values in the three assignments are T, F, T respectively, then its value in A_4 is T because T occurs twice while F only once. Hence, roughly speaking, $A_4 = \text{majority}(A_1, A_2, A_3)$. Verify that A_4 is again a truth assignment that satisfies C .
- c. Show that in fact part (b) is true for an arbitrary instance of 2-SAT. That is, consider an arbitrary instance C of 2-SAT constructed from the variables x_1, x_2, \dots, x_n . Suppose that A_1, A_2 and A_3 are three truth assignments of the variables that satisfy C . Show that $A_4 = \text{majority}(A_1, A_2, A_3)$ is also a truth assignment that satisfies C . (Hint: Prove by contradiction.)
 - c1. Okay, here's some help. Suppose A_4 does not satisfy C . Now, C is made up of clauses combined with “and” operators. What should at least one of the clauses in C evaluate to?
 - c2. If a clause evaluates to false, what should the literals inside it evaluate to?

Now, you're on your own...